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THE DIDACTICAL USE OF MODELS IN REALISTIC MATHEMATICS EDUCATION: AN EXAMPLE FROM A LONGITUDINAL TRAJECTORY ON PERCENTAGE[1]

ABSTRACT. The purpose of this article is to describe how, within the Dutch approach to mathematics education, called *Realistic Mathematics Education* (RME), models are used to elicit students' growth in understanding of mathematics. First some background information is given about the characteristics of RME related to the role of models in this approach. Then the focus is on the use of the bar model within a longitudinal trajectory on percentage that has been designed for *Mathematics in Context*, a curriculum for the U.S. middle school. The power of this model is that it develops alongside both the teaching and the students: from a drawing that represents a context related to percentage to a strip for estimation and reasoning to an abstract tool that supports the use of percentage as an operator.

KEY WORDS: contexts, curriculum design, mathematics education, models, percentage, primary school, shift in levels of understanding

INTRODUCTION

Realistic Mathematics Education (RME) is a domain-specific instruction theory for mathematics education (e.g., Treffers, 1987; De Lange, 1987; Streefland, 1991, Gravemeijer, 1994a; Van den Heuvel-Panhuizen, 1996). This theory is the Dutch answer to the need, felt worldwide, to reform the teaching of mathematics. The roots of RME go back to the early 1970s when Freudenthal and his colleagues laid the foundations for it at the former IOWO[2], the earliest predecessor of the Freudenthal Institute. Based on Freudenthal's (1977) idea that mathematics – in order to be of human value – must be connected to reality, stay close to children and should be relevant to society, the use of realistic contexts became one of the determining characteristics of this approach to mathematics education. In RME, students should learn mathematics by developing and applying mathematical concepts and tools in daily-life problem situations that make sense to them.

On the one hand the adjective 'realistic' is definitely in agreement with how the teaching and learning of mathematics is seen within RME, but on the other hand this term is also confusing. In Dutch, the verb 'zich

Educational Studies in Mathematics **54:** 9–35, 2003.

realiseren' means 'to imagine'. In other words, the term 'realistic' refers more to the intention that students should be offered problem situations which they can *imagine* (see Van den Brink, 1973; Wijdeveld, 1980) than that it refers to the 'realness' or authenticity of problems. However, the latter does not mean that the connection to real life is not important. It only implies that the contexts are not necessarily restricted to real-world situations. The fantasy world of fairy tales and even the formal world of mathematics can be very suitable contexts for problems, as long as they are 'real' in the students' minds.

Apart from this often-arising misconception about the meaning of 'realistic' the use of this adjective to define a particular approach to mathematics education has an additional 'shortcoming'. It does not reflect another essential feature of RME: the didactical use of models. In this article the focus will be on this aspect of RME.

In the first part of this position paper I will give general background information about the theory of RME and the role of models within this theory. Among other things, attention will be paid to the two ways of mathematizing that characterize RME, the different levels of understanding that can be distinguished and that typify the learning process, the way students can play an active role in developing models and how models can evolve during the teaching-learning process, and – as a result of this – can prompt and support level raising. In the second part of the article this general information will be made more concrete by concentrating on the content domain of percentage. A description is given of how the bar model can support the longitudinal process of learning percentage.

This description of the didactical use of the bar model is based on the development work carried out in the *Mathematics in Context* project, a project aimed at the development of a mathematics curriculum for the U.S. middle school (Romberg, 1997–1998). The project was funded by the National Science Foundation and executed by the Center for Research in Mathematical Sciences Education at the University of Wisconsin-Madison[3], and the Freudenthal Institute of Utrecht University. The designed curriculum reflects the mathematical content and teaching methods suggested by the 'Curriculum and Evaluation Standards for School Mathematics' (NCTM, 1989). This means that the philosophy of the curriculum and its development is based on the belief that mathematics, like any other body of knowledge, is the product of human inventiveness and social activities. This philosophy has much in common with RME. It was Freudenthal's (1987) belief that mathematical structures are not a fixed datum, but that they emerge from reality and expand continuously in individual and collective learning processes. In other words, in RME students are seen as active participants in the teaching-learning process that takes place within the social context of the classroom.

In addition to the foregoing, however, Freudenthal (1991) also emphasized that the process of re-invention should be a guided one. Students should be offered a learning environment in which they can construct mathematical knowledge and have possibilities of coming to higher levels of comprehension. This implies that scenarios should be developed that have the potential to elicit this growth in understanding. The development of such a scenario for learning percentage was one of the goals of the *Mathematics in Context* project. Within this scenario the bar model was the main didactical tool to facilitate the students' learning process.

RME AND THE DIDACTICAL USE OF MODELS

Mathematics as mathematizing

One of the basic concepts of RME is Freudenthal's (1971) idea of mathematics as a human activity. As has been said before, for him mathematics was not the body of mathematical knowledge, but the activity of solving problems and looking for problems, and, more generally, the activity of organizing matter from reality or mathematical matter – which he called 'mathematization' (Freudenthal, 1968). In very clear terms he clarified what mathematics is about: "There is no mathematics without mathematizing" (Freudenthal, 1973, p. 134).

This activity-based interpretation of mathematics had also important consequences for how mathematics *education* was conceptualized. More precisely, it affected both the goals of mathematics education and the teaching methods. According to Freudenthal, mathematics can best be learned by doing (ibid., 1968, 1971, 1973) and mathematizing is the core goal of mathematics education:

> What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics. (Freudenthal, 1968, p. 7)

Although Freudenthal in his early writings unmistakably referred to two kinds of mathematizing, and he made it clear that he did not want to limit mathematizing to an activity on the bottom level, where it is applied to organize unmathematical matter in a mathematical way, his primary focus was on mathematizing reality in the common sense meaning of the world out there. He was against cutting off mathematics from real-world situations and teaching ready-made axiomatics (Freudenthal, 1973).

Two ways of mathematizing

It was Treffers (1978, 1987) who placed the two ways of mathematizing in a new perspective, which caused Freudenthal to change his mind as well. Treffers formulated the idea of two ways of mathematizing in an educational context. He distinguished 'horizontal' and 'vertical' mathematizing. Generally speaking the meaning of these two forms of mathematizing is the following. In the case of horizontal mathematizing, mathematical tools are brought forward and used to organize and solve a problem situated in daily life. Vertical mathematizing, on the contrary, stands for all kinds of re-organizations and operations done by the students within the mathematical system itself. In his last book Freudenthal (1991) adopted Treffers' distinction of these two ways of mathematizing, and expressed their meanings as follows: to mathematize horizontally means to go from the world of life to the world of symbols; and to mathematize vertically means to move within the world of symbols. The latter implies, for instance, making shortcuts and discovering connections between concepts and strategies and making use of these findings. Freudenthal emphasized, however, that the differences between these two worlds are far from clear cut, and that, in his view, the worlds are not, in fact, separate. Moreover, he found the two forms of mathematizing to be of equal value, and stressed the fact that both activities could take place on all levels of mathematical activity. In other words, even on the level of counting activities, for example, both forms may occur.

Although Freudenthal introduced some important nuances in the formulation of the two ways of mathematizing, these do not affect the core of Treffer's classification or its significance. Furthermore, it was Treffers' merit that he made it clear that RME clearly differentiates itself, through this focus on two ways of mathematizing, from other (then prevailing) approaches to mathematics education. According to Treffers (1978, 1987, 1991) an empiricist approach only focuses on horizontal mathematizing, while a structuralist approach confines oneself to vertical mathematizing, and in a mechanistic approach both forms are missing. As Treffers and Goffree (1985) stressed, the kind of mathematizing on which one is focused in mathematics education has important consequences for the role of models in the different approaches to mathematics education, and also for the kind of models that are used.

Different levels of understanding

Another characteristic of RME that is closely related to mathematizing is what could be called the 'level principle' of RME. Students pass through different levels of understanding on which mathematizing can take place:

from devising informal context-connected solutions to reaching some level of schematization, and finally having insight into the general principles behind a problem and being able to see the overall picture. Essential for this level theory of learning – which Freudenthal derived from the observations and ideas of the Van Hieles (see, for instance, Freudenthal 1973, 1991) – is that the activity of mathematizing on a lower level can be the subject of inquiry on a higher level. This means that the organizing activities that have been carried out initially in an informal way, later, as a result of reflection, become more formal.

This level theory of learning is also reflected in 'progressive mathematization' that is considered as the most general characteristic of RME and where *models* – interpreted broadly – are seen as vehicles to elicit and support this progress (Treffers and Goffree, 1985; Treffers, 1987; Gravemeijer, 1994a; Van den Heuvel-Panhuizen, 1995, 2002). Models are attributed the role of bridging the gap between the informal understanding connected to the 'real' and imagined reality on the one side, and the understanding of formal systems on the other.

Broad interpretation of models

Within RME, models are seen as representations of problem situations, which necessarily reflect essential aspects of mathematical concepts and structures that are relevant for the problem situation, but that can have different manifestations. This means that the term 'model' is not taken in a very literal way. Materials, visual sketches, paradigmatic situations, schemes, diagrams and even symbols can serve as models (see Treffers and Goffree, 1985; Treffers 1987, 1991; Gravemeijer 1994a). For instance, an example of a paradigmatic situation that can function as a model, is repeated subtraction. Within the learning strand on long division, this procedure – elicited, for instance, by the transit of a large number of supporters by coach (see Gravemeijer 1982; Treffers, 1991) – both legitimizes and gives access to the formal long division algorithm. As an example of a way of notation the arrow language can be mentioned. The initial way of describing the changes in the number of passengers on a bus ends up being used to describe all kind of numerical changes later on (see Van den Brink, 1984).

For being suitable to give the intended support to learning processes, models must have at least two important characteristics. On the one hand they have to be rooted in realistic, imaginable contexts and on the other hand they have to be sufficiently flexible to be applied also on a more advanced, or more general level. This implies that a model should support progression in vertical mathematizing without blocking the way back to the sources from which a strategy originates – which is similar to the Vygotskian notion of scaffolding (Vygotsky, 1978). In other words, the students should always be able to revert to a lower level. It is this two-way character of models that makes them so powerful. Another requirement for models to be viable is that they – in alignment with the RME view of students as active participants of the teaching-learning process – can be re-invented by the students on their own. To realize this, the models should 'behave' in a natural, self-evident way. They should fit with the students' informal strategies – as if they could have been invented by them – and should be easily adapted to new situations.

A closer look at the level raising power of models

Coming to the point of why models can contribute to level raising, the work of Streefland comes into the picture. About fifteen years ago, Streefland (1985a) elucidated in a Dutch article how models can fulfill the bridging function between the informal and the formal level: by shifting from a *'model of'* to a *'model for'*. In brief, this means that in the beginning of a particular learning process a model is constituted in very close connection to the problem situation at hand, and that later on the context-specific model is generalized over situations and becomes then a model that can be used to organize related and new problem situations and to reason mathematically. In that second stage, the strategies that are applied to solve a problem are no longer related to that specific situation, but reflect a more general point of view. In the mental shift from 'after-image' to 'pre-image' the awareness of the problem situation and the increase in level of understanding become manifest.[4] The change of perspective involves both insight into the broader applicability of the constructed model, and reflection on what was done before (Streefland, 1985a; see also 1992, 1993, 1996). Especially in the areas of fractions, ratio and percentage Streefland enriched the didactics of mathematics education with models that have this shifting quality.

A first example is connected to his design research on fractions within the context of a pizza restaurant (Streefland, 1988, 1991). In the trajectory he designed, the learning process starts with the 'concrete' model of the 'seating arrangements'[5] to compare amounts of pizza, which model is evoked by the designed tasks that are presented to the students, and later schematized to the 'seating arrangement tree' and the ratio table by means of which formal fractions are compared and operations with fractions are carried out. In this process of schematization and generalization, again the roles of the designer and the teacher are very important. By designing a trajectory in which new problems prompt the students to arrive at adaptations of the initial 'concrete' model and by accentuating particular adaptations that the students come up with the process of model development is guided.

The bar model that will be discussed later in this article is a second example. In the development of teaching a unit on percentage in which this bar model is the backbone for progress, Leen Streefland and I worked very closely together.

Although we owe the concept of the shifts in models to Streefland, he did not do his work in isolation. Again, the role Freudenthal played should not be underestimated. The distinction between the two meanings of 'model' was already an issue in his writings in the 1970s, when he wrote: "Models *of* something are after-images of a piece of given reality; models *for* something are pre-images for a piece of to be created reality" (Freudenthal, 1975, p. 6[6]). In connection with these two functions of models he distinguished also 'descriptive models' and 'normative models' (Freudenthal, 1978). However, the difference with Streefland is that Freudenthal was thinking about models at a much more general didactical level – such as models for lessons, curriculum plans, goal descriptions, innovation strategies, interaction methods, and evaluation procedures – and not on the micro-didactical level that Streefland had in mind. By applying Freudenthal's thinking within a micro-didactic context he revealed the level raising mechanisms of models and the didactical use of this power. His idea of 'model of' and 'model for' undoubtedly turned out to be an eye-opener for many (see e.g., Treffers, 1991; Gravemeijer, 1994a, 1994b, 1997, 1999; Van den Heuvel-Panhuizen, 1995, 2001; Gravemeijer and Doorman, 1999; Yackel et al., 2001, Van Amerom, 2002). It is a simple, immediately recognizable and applicable idea, in which the essence of learning processes, namely raising the level of knowledge, is given a didactical entrance. For this reason it has been followed up in thinking about the didactics of mathematics education both within and without the RME community.

In particular, Gravemeijer (1994a, 1994b, 1997, 1999) worked out this idea. He showed that the shift in models can also be connected to the process of mathematical growth in a more general way. The distinction between 'model of' and 'model for' led him to split up the intermediate level, between the situational level and the formal level of solving problems and mathematical understanding, into a referential and a general level. In addition to this, Gravemeijer emphasized the connection between the use of models and the re-invention principle of RME. Because of the shift in model – that causes the formal level of mathematics to become linked to informal strategies – the top-down element that characterized the use of models within the structuralist and cognitive approaches to mathematics education could be converted to a bottom-up process.

How to find suitable models and model-eliciting activities?

Although the bottom-up process implies that the models are invented by the students themselves, the students should be provided with a learning environment – the whole of problems, activities, and contexts, placed within scenarios or trajectories, together with the stimulating and accentuating role of the teacher – to make this happen. As said earlier, within RME, re-invention is taken to be guided re-invention. However, an essential facet of this process is that the students should have the feeling of having the lead in it. The emergence of models and their further evolution must occur in a natural manner.

The previous requirement puts a large onus on the development of educational materials. Education developers have to look for problem situations that are suitable for model building and fit within a scenario or trajectory that elicits the further evolution of the model, to let it grow into a didactic model that opens up the path to higher levels of understanding for the students. It should be clear that this puts certain demands on such a problem situation. A key requirement is that the problem situation can be easily schematized. Another demand is that, from the point of view of the students, there should be a necessity for model building. This aspect requires that the problem has to include model-eliciting activities, such as for instance, planning and executing solutions steps, generating explanations, identifying similarities and differences, and making predictions. Although these criteria already give a good indication of what is necessary to have a model emerge, the most important is that the problem situations and activities bring the students to identify mathematical structures and concepts. To discover which problems and activities can do this, 'phenomenological didactical analyses', as Freudenthal (1978, 1983) called them, are needed. These analyses are focused on how mathematical knowledge and concepts can manifest themselves to students and how they can be constituted. Part of this analysis is done by means of thought experiments and intercolleague deliberation – including discussions with teachers – in which both knowledge about students and ideas about the desired mathematical concepts function as a guiding pre-image. The more important part of the analysis, however, is done while working with students and analyzing students' work. In this way what is important for constituting the model and hence what has to be 'put' in the problem situation can be found, so that situation-specific solutions can be elicited, which can be schematized, and which will have vertical perspective.

THE BAR MODEL FOR LEARNING PERCENTAGE AS AN EXAMPLE

In the remaining part of this article the didactical use of models in RME is illustrated by the use of the bar model in a longitudinal learning-teaching trajectory on percentage that was designed for the *Mathematics in Context* curriculum. Simply put, this bar model refers to a strip on which different scales are depicted at the same time, as a result of which an amount or a quantity can be expressed through a different amount or quantity. Through this, the bar model touches on the essence of a rational number such as percentage.

The main part of the percentage trajectory extends over three teaching units of this curriculum:

- *Per Sense* (Van den Heuvel-Panhuizen et al., 1997), is meant for grade 5 and intends to be a starting unit on percentage;
- *Fraction Times* (Keijzer et al., 1998b), is meant for grade 6 and covers the domain of rational number more broadly and contains material about percentages, fractions, decimals and ratios;
- *More or Less* (Keijzer et al., 1998a), is meant for grade 6 and focuses on percentages, fractions and decimals.

Because my focus in this article is on giving a view of the longitudinal trajectory and connections within it, I will restrict myself to the learning of percentage. The conclusion that, within *Mathematics in Context,* the teaching of percentage is considered a separate teaching strand should not be drawn, however. On the contrary, learning percentage is embedded within the whole of the rational number domain and is strongly entwined with learning fractions, decimals and ratios with the bar model connecting these rational number concepts (see Middleton, Van den Heuvel-Panhuizen, and Shew, 1998). However, the bar model is not the only supporting model for this domain. Apart from the bar, which later becomes a double number line, the ratio table and the pie-chart also play an important role in the *Mathematics in Context* trajectory on percentage (see Wijers and Van Galen, 1995; Middleton and Van den Heuvel-Panhuizen, 1995). For the sake of clarity, this article will avoid describing the complexity that is typical in this learning process. Nor will attention be given to how the trajectory on percentage was developed and how the bar model found its place within the trajectory. Regarding the *Per Sense* unit, information about this design process can be found in Van den Heuvel-Panhuizen and Streefland (1993). The assessment that was developed for this unit is reported in Van den Heuvel-Panhuizen (1994, 1996).

The purpose of this article is to describe how, within a series of teaching units as designed for the *Mathematics in Context* curriculum, the bar model emerges and evolves, and supports the students' learning. The description is based on snapshots taken from the draft versions of these units[7], including some student work that shows to what degree the intended process of model building is in line with the students' ways of working and thinking. The latter is important because it enables them to re-invent the model on their own, or at least, to participate actively in the process of model building.

A brief overview of the percentage learning-teaching trajectory

In the three *Mathematics in Context* units the learning-teaching trajectory on percentage starts with a qualitative way of working, with percentages as descriptors of so-many-out-of-so-many situations, and ends with a more quantitative way of working with percentages by using them as operators. During this process of growing understanding of percentage, the bar gradually changes from a concrete context-connected representation to a more abstract representational model that moreover is going to function as an estimation model, and to a model that guides the students in choosing the calculations that have to be made. This means that the model then becomes a calculation model. At the end of the trajectory, when the problems become more complex, it can also be used as a thought model for getting a grip on problem situations. However, the foregoing does not mean that separate stages in the use of the bar model can be distinguished, or that there is a strict order in which these different applications are learned; this is not the case. Indeed, though there is a kind of sequence laid down in the teaching units, the different interpretations of the bar model are accessible in all stages of the learning process. It all depends on how the students see and use the model.

Another change to the bar has to do with its form. Together with the change in function the appearance of the model changes. Eventually the bar is reduced to a double number line. Although there is not a large difference between these two models – both can be seen as a strip on which on either side different units of measurement are used – this change has the advantage that the bar model becomes simpler and thus easier to use, and that it becomes more flexible. Among other things, this change makes the model more suitable for going beyond one hundred percent.

Some first explorative activities

With the point of departure that education must build on the informal, preschool and outside school knowledge of the students, in mind, the *Per Sense* unit – aimed at having the students making sense of percentage – starts with an introductory chapter in which the students are confronted

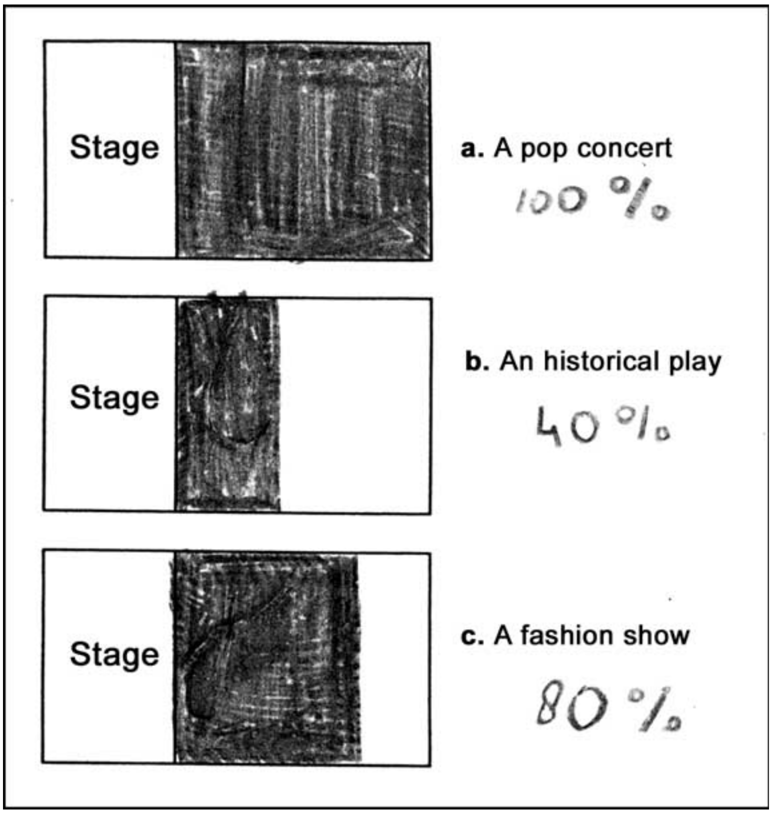

Figure 1. Percentage of occupied seats in school theater.

with some daily-life stories in which percentages play a role. A more extensive description of what these stories can reveal about the students' informal knowledge on percentages can be found in Streefland and Van den Heuvel-Panhuizen (1992).

One of the stories is about a boy who is telling his mother that there is a ninety five percent chance that soccer practice will remain on Wednesdays. Besides discussing the qualitative meaning of this ("95 percent means that it is almost sure that..."), the students are also asked to use drawings in explaining this meaning. In this way this first chapter includes some explorative activities that prepare model building. A special role regarding this has been reserved for several assignments based on the school theater. The students are asked to indicate for different performances how busy the theater will be. They can do this by coloring in the part of the hall that is occupied and then writing down the percentage of the seats that are occupied (see Figure 1).

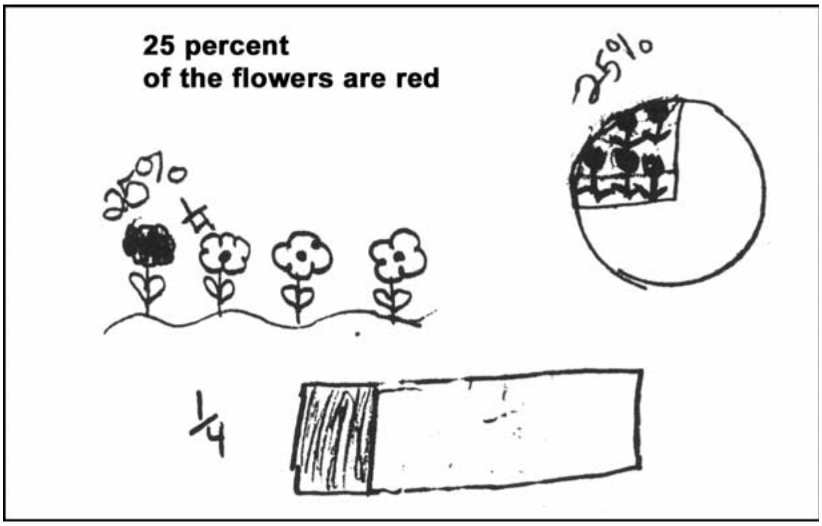

Figure 2. The use of drawings to express percentages.

It was remarkable how easily the children got to work on this assignment. There were practically no questions. Everything happened very naturally, and it was clear from the way in which the children discussed the different performances that they knew what the percentages represented. In the case of the historical play “the theater hall was less than half full” and “you could easily choose where you wanted to sit”.

In the same way as in the theater task, in a summarizing activity at the end of the first chapter the students are asked to use drawings to express what is said in particular statements including percentages. As is shown in Figure 2, the students came up spontaneously with all kinds of models, ranging from pictorial drawings to pie charts and even bars.

Observations during the try-outs of the teaching unit showed that the set-up provoking the use of bars with the school theater activity worked. For the students, this coloring in of theater halls also became a way to express other kinds of so-many-out-of-so-many situations. Here, in other words, a first shift from a ‘model of’ to a ‘model for’ is made. Another interesting finding was that the students also spontaneously used fractions to ‘explain’ the percentage of fullness. This means that the awareness of this connection between different rational numbers, which actually is one of the final goals to achieve at the general, formal level, is in essence already present at the context-connected, informal level of understanding.

The next chapter of the *Per Sense* unit includes a set of problems in the context of parking. The students are asked to compare parking lots with respect to their fullness. Again, the students are asked to indicate the

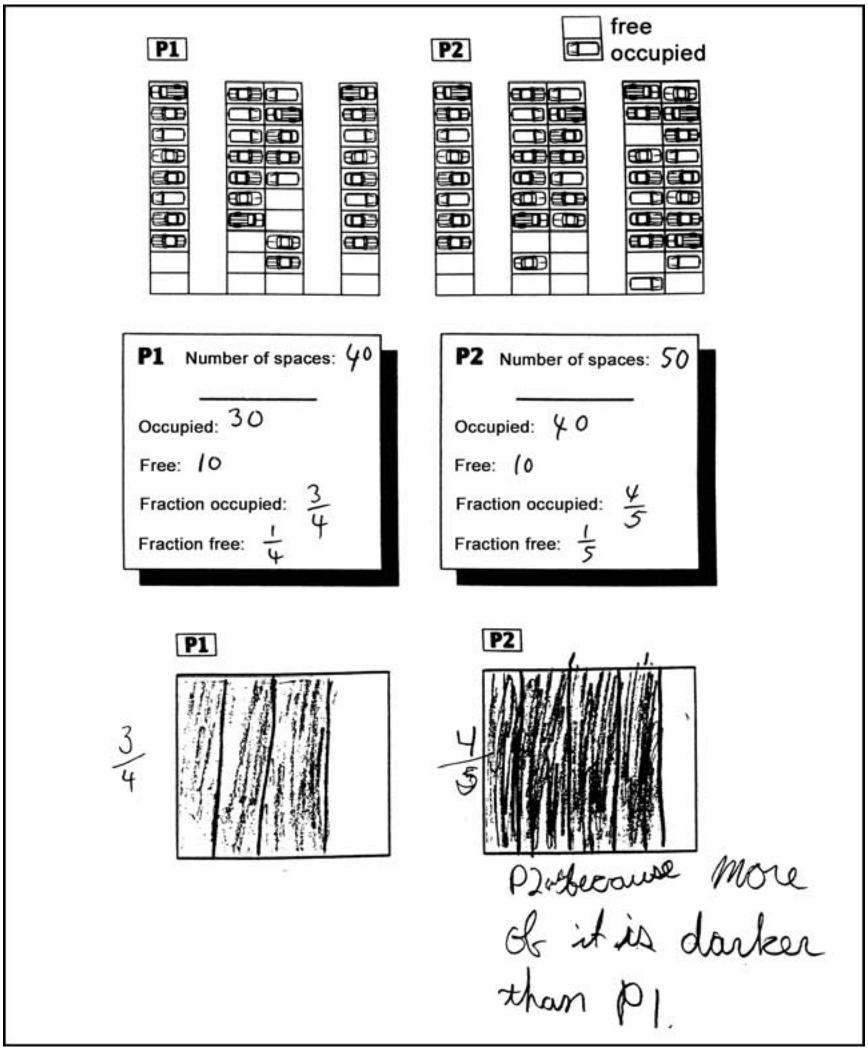

Figure 3. Comparing the fullness of parking lots.

degree of occupation for each parking lot by coloring in the rectangular frame that represents the parking lot. Next it can be determined which parking lot is the fullest (see Figure 3).

The emergence of the bar model

The following step is that the rectangular frame that represents the 'real' parking lot is replaced by an 'occupation meter'. Such a meter is similar to, for instance, a display to check the amount of dust in a vacuum cleaner or a charge indicator for batteries. Like these, the occupation meter offers

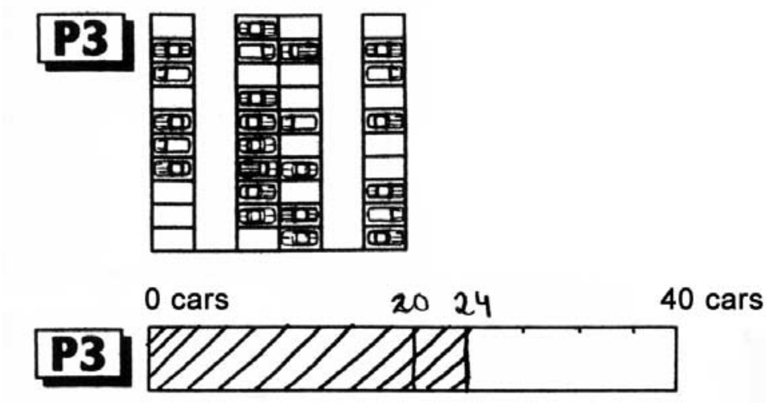

Figure 4. The 'occupation meter' shows the fullness of the parking lot.

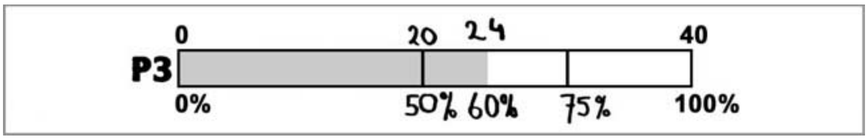

Figure 5. The 'occupation meter' reveals the percentage of fullness of the parking lot.

the students a way to represent the parking lot's fullness. They can again color in the part that is occupied (see Figure 4).

Moreover, after doing this, the 'occupation meter' visualizes the percentage of occupied spaces for the students. If the meter is completely colored in, it means that the parking lot is 100% full. If 24 out of 40 spaces are occupied the parking lot is filled for, let us say as a preliminary first answer, a little bit over 50%. But after indicating 75% as the middle between 50% and 100%, and using it as a reference, 60% might come up as 'a good guess' (see Figure 5).

Depending on the actual numbers in these parking lot problems, the occupation meter can be used in different ways to find the percentage of fullness (see Figure 6). If 60 spaces out of 80 are occupied (a), the students can make use of an easy fraction. In the case of 50 spaces out of 85 (b), the percentage of fullness can be approximated by a strategy based on repeated halving. And finally, when the figures are 36 out of 40 (c) the students can making use of a known percentage, 10% of 40 is 4, thus... (see Figure 6).

In other words, there is no fixed strategy to solve these percentage problems, and the occupation meter allows this flexibility in approach. There is another great advantage to such an approach, next to the didactic advantage of being able to connect flexibly to the differences in the students' knowledge of numbers – the benchmarks and number relations the students have on hand. Using this approach makes it possible that what is the aim at

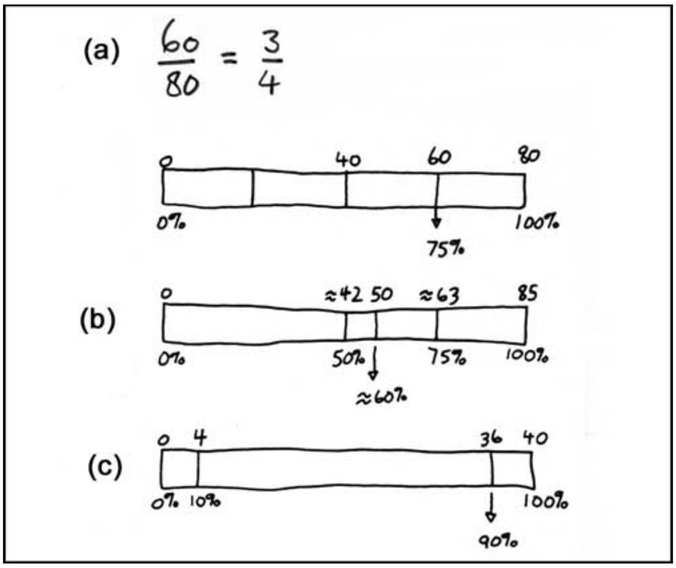

Figure 6. Different ways of using the 'occupation meter' for finding the percentage of fullness.

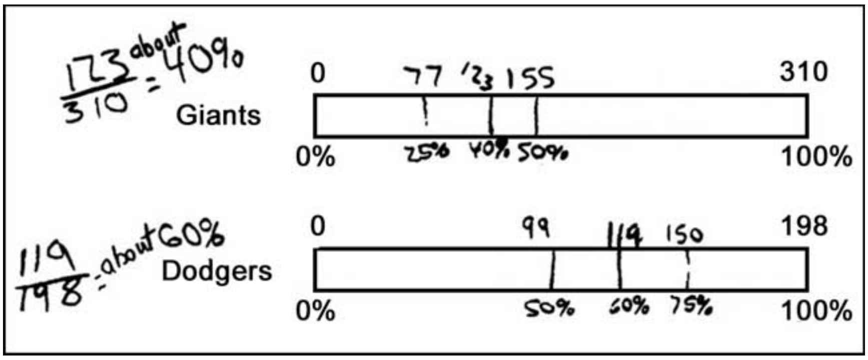

Figure 7. Using the bar as an estimation model.

the highest level – making handy and flexible use of networks of numbers and properties of relations and operations – is already elicited at the lowest level.

The bar as an estimation model

Later on, in chapter three of the *Per Sense* unit, the 'occupation meter' gradually changes into a plain bar model. In other words, again a shift is made from a 'model of' to a 'model for' – that is to say from the perspective of teaching; the real shift, of course, is made in the students' thinking. The shift means that the model is no longer exclusively connected to the

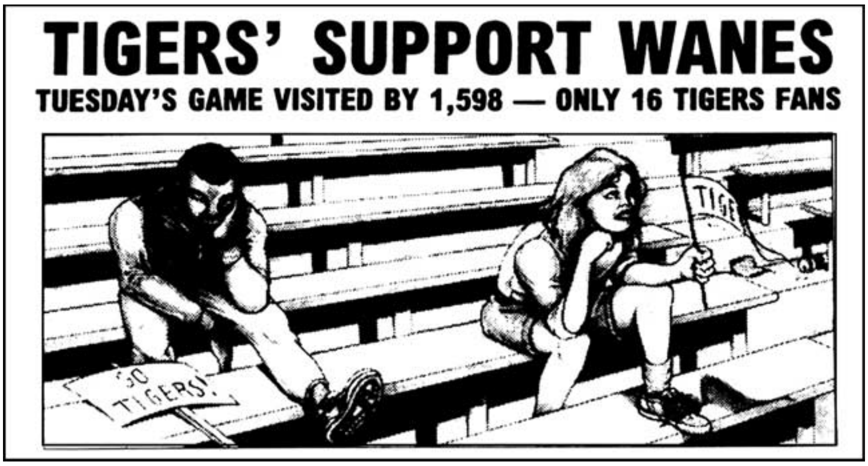

Figure 8. Introduction of 1% as a benchmark.

parking lot context, but that it helps the students, for instance, to compare the preference of fans for particular baseball souvenirs. Moreover, the shift gives access to a higher level of understanding, in which the bar is used to reason about so-many-out-of-so-many situations. Especially in cases where the problems concern numbers that cannot be simply converted to an easy fraction or percentage, the bar gives a good hold for estimating an approximate percentage. An example of this is shown in Figure 7. The problem is about two groups of fans, the Giants fans and the Dodgers fans, who have been interviewed about their favorite baseball souvenir. In total, 310 Giant fans have been interviewed and 123 of them chose the cap as their favorite souvenir. In the case of the Dodgers fans, 119 out of 198 fans chose the cap. The students are asked which fans like the cap the best?

In order to provide the students with a more precise strategy, later on in this chapter their attention is also drawn to the 1%-benchmark. This is done more or less casually through a headline in the newspaper, which is about a very low attendance of Tigers fans (see Figure 8).

The bar as a calculation model

This 1%-benchmark is introduced to open the way to calculate percentages, but the approach chosen in this trajectory is rather different from the usual way of making precise calculation using 1%; it is used for calculating percentages in an approximate way. It should not be confused with calculating precisely by means of a calculator, which comes later. In contrast with this, using 1% as a benchmark here is still a form of estimating. The difference with the form of estimation which was dealt with in the previous paragraph is that now the bar itself is not used to operate on, but is only

Year of marathon	Total number of competitors	Number of dropouts	Percent of dropouts	Describe your strategy
1991	1,603	91	≈5%	≈16 1,603 1% 5 16)91 -80 11

Figure 9. Using the bar as a calculation model with 1% as a benchmark.

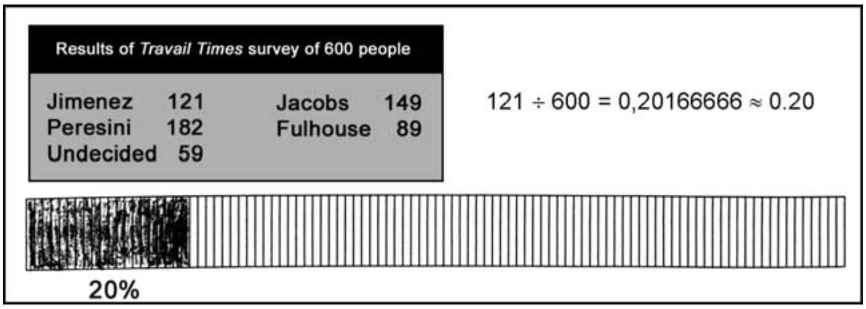

Figure 10. A so-many-out-of-so-many situation converted into a percentage via a decimal.

used to guide the students in calculating the percentage. The bar tells them in an understandable manner what calculation they have to carry out to find the answer (see Figure 9).

The bar is also relevant for the reverse, though, since it can also give an insight into the results of calculations, which can be important for understanding the relationship between percentages and decimal numbers. This is particularly important when working with percentages as operators.

A first step to this next stage in the learning of percentage is made in the grade 6 unit *Fraction Times*[8] where the students learn to convert so-many-out-of-so-many situations ‘directly’ into a percentage. Instead of dividing the part by 1% of the total number, now the part is divided directly by the total number. This latter strategy clearly gives a different result than the first one, but it does not effect the ratio between the part and the whole, as the students have experienced in their working with the ratio table which has a very central role in this teaching unit. As a result of working with the ratio table, the students gradually learn to interpret the ratio in a flexible way, they can work towards a so-many-out-of-hundred situation and discover also that they can replace this so-many-out-of-hundred situation with a so-many-out-of-thousand, out-of-ten or out-of-one situation. Such experiences in their turn, help in interpreting the decimal answer of the

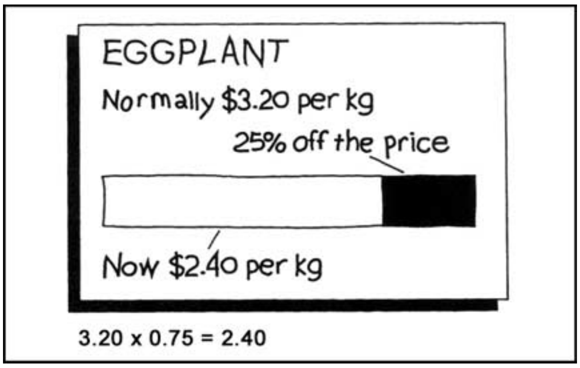

Figure 11. Checking the sale price by one multiplication.

division in which the part is divided directly by the total number, as an answer that stands for a so-many-out-of-ten, hundred, thousand, etcetera expression that can be depicted on a bar. Depending on the degree of accuracy needed, any bar is suitable for this, but in the case of expressing the ratio as a percentage, the 100-segment bar is most suitable, as is done for instance in the problem about the results of an election (see Figure 10).

The percentage of votes that Jiminez got in the election is found by dividing his number of votes by the total number of respondents. The decimal that appears on the display of the calculator tells how many segments out of the hundred have to be colored in. At the same time it is still possible to make an estimation: 121 out of 600 is approximately one fifth of the total, or about 20%.

Later in grade 6, in the *More or Less* unit, the students are confronted with situations of change. Then they learn to express – both in an additive (+25% or –25%) and in a multiplicative way (× 1.25 or × 0.75) – new situations as a percentage of the old ones. This part of the trajectory starts with a situation of price reduction. The example that is shown in Figure 11 is about a supermarket that introduced new price tags. The students are asked to check the sale price by making only one multiplication on their calculator.

After this short introduction connected to prices, percentages as operators are further explored in the final chapter of the *More or Less* unit. The chapter starts with the context of a photocopier that can reduce and enlarge. The copier's maximum reduction option is 80%. Among other things, the students are confronted with a situation in which one reduction of 80% is not enough and multiple reductions of 80% are needed. Connected to the calculation of the effect of a double reduction on the dimensions of a picture, an elastic strip[9] is used to make an estimation of the result of the double reduction (see Figure 12).

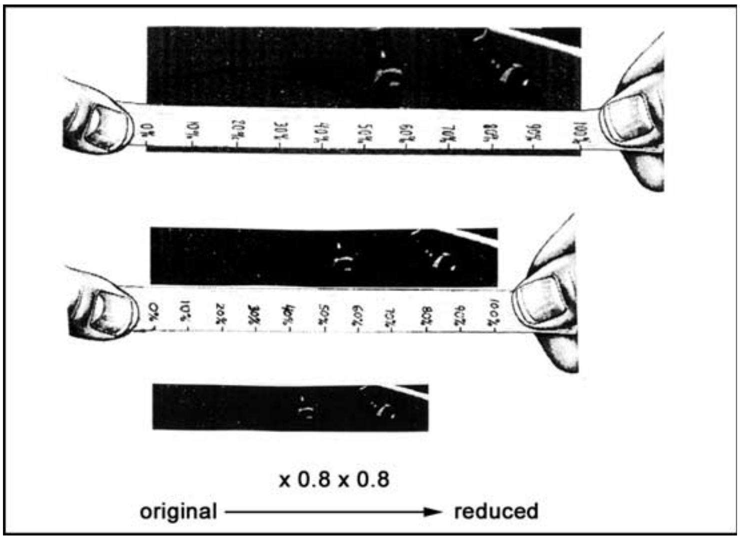

Figure 12. Using an elastic strip to find the result of double reduction of 80%.

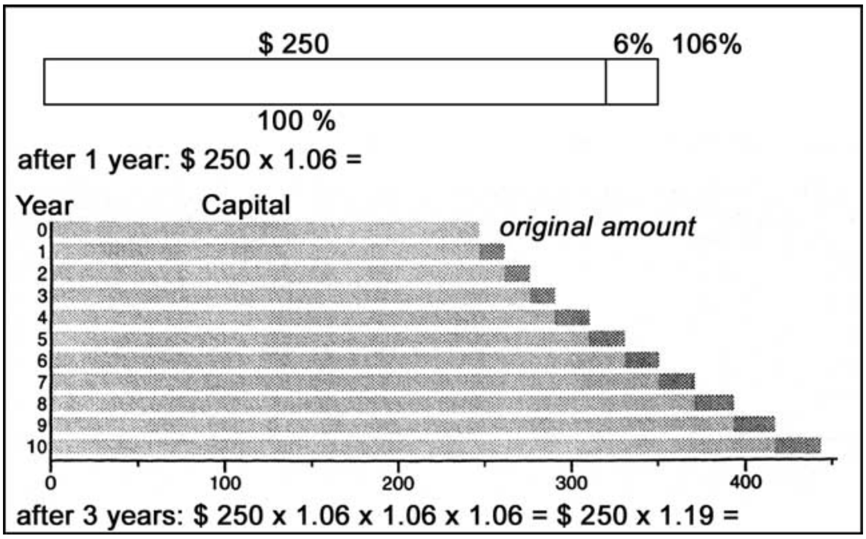

Figure 13. Bar graph showing how the money grows in an interest-bearing savings account.

Later on, exponential increases – though they are not referred to as such to the students – are dealt with in the context of an interest-bearing savings account. Again the bar can make visible how this works and what calculation one has to do in order to find the total amount of money after one year, two years, three years, and so on (see Figure 13).

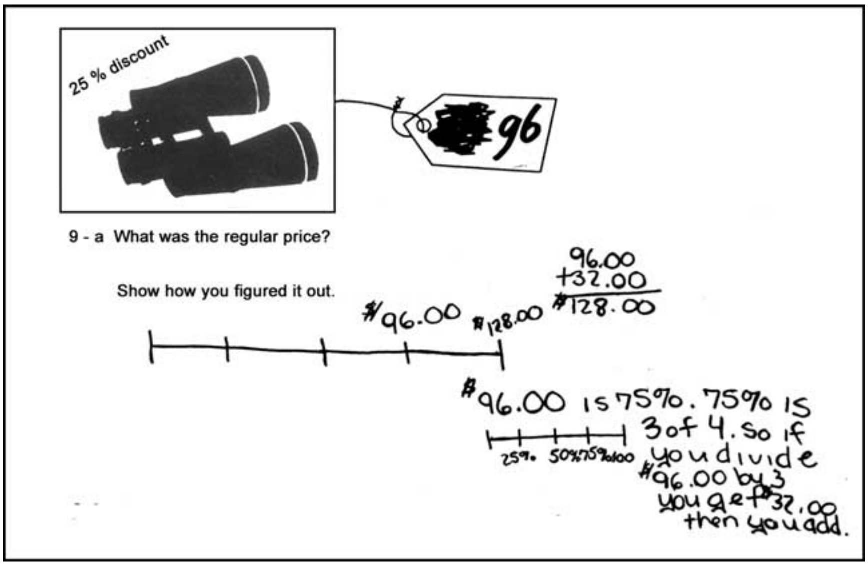

Figure 14. The double number line as a support for backwards reasoning.

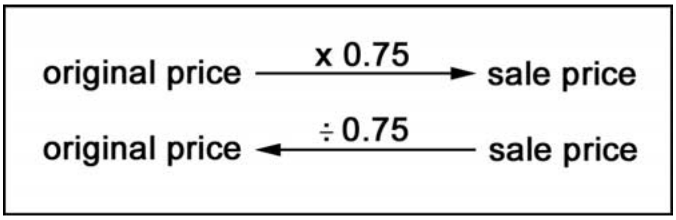

Figure 15. Finding the original price as the reverse of finding the sale price.

The bar as a thought model

As is shown in the previous example, the bar can also be helpful in understanding complex situations. The same applies to situations that ask for backward reasoning, which is the case in the next problem. Here the sale price and the discount percentage are given and the students have to find out the original price (see Figure 14).

In the student work shown in Figure 14, instead of the bar a simple double number line is used to support the backwards reasoning. It confirms in a way the natural switch from one version of the model to another. Crucial for both versions is that they help the students to understand that the sale price equals 75% of the original price and that they have to divide the sale price by 3 and then multiply it by 4.

On a higher level, however, the original price can be found by means of a one-step division by dividing the sale price by three fourths or by seventy-five hundredths, which is the opposite of finding the sale price when the original price and the percentage of discount has been given (see Figure 15).

Actually, this latter solution is an example of vertical mathematizing. It is based on a shortcut within the mathematical system.

TO CONCLUDE: A REFLECTION ON THE DIDACTICAL USE OF MODELS

The foregoing snapshots from a learning-teaching trajectory on percentage illustrate how, within RME, models are used as didactical tools for teaching mathematics. The didactical perspective taken in this article means that the spotlight here was not on modeling as a goal of mathematics education, although this is of course a significant characteristic of RME which at the same time characterizes recent thinking about mathematics education. A good example of the latter is the work of Lesh and Doerr (2000). Modeling, in their interpretation, relates to the process of model development through which students gradually gain better understanding of a rich, meaningful problem situation by describing and analyzing it with more and more advanced means, and by going through a series of modeling cycles they finally develop an effective model with which they can also take on other (similar) complex problem situations. The focus in this article, however, was not on how students can be taught to solve problems through building models by progressive mathematizing, but on how mathematical concepts such as rational number and especially the understanding of percentage can be taught. Although both learning processes are necessary, have a lot in common and support each other, working on students' model building attitude will not be enough to teach them percentage.

This article focused rather on how they can learn percentage and how models can be used didactically to realize this learning process.

Formulated more precisely, it is not the models in themselves that make the growth in mathematical understanding possible, but the students' modeling activities. Within RME, students are not handed ready-made models that embody particular mathematical concepts, but they are confronted with context problems, presented in such a way that they elicit modeling activities, which in their turn lead to the emergence of models. Moreover, the longitudinal perspective of the percentage trajectory demonstrates clearly that the model that emerges here, the bar model, develops more and more throughout the trajectory. The initial modeling activities, executed on context problems embedded in the students' reality, accomplish that the students arrive at new realities, which in their turn can become the subject of new modeling activities. This shift results in the bar model manifesting its function in different ways at different points in the tra-

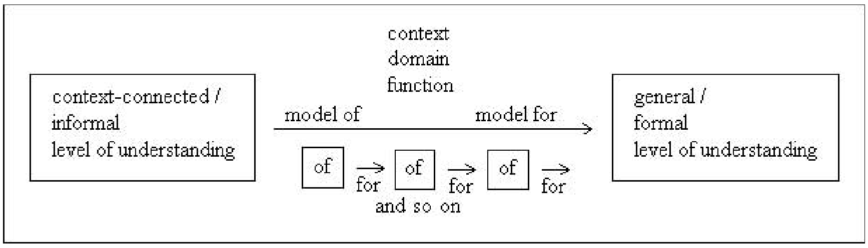

Figure 16. Levels of understanding and the shifts from 'model of' to 'model for'.

jectory: from a picture of a so-many-out-of-so-many situation to an occupation meter to a double number line. In fact, the modeling activities do not produce one single model, but a chain of models. Evoked by a sequence of problems presented in a learning environment that stimulates reflection and classroom interaction, new manifestations of the model keep coming into view, giving access to new perspectives, new possibilities for problem solving and higher levels of understanding, but at the same time encompassing previous manifestations of the model. All this implies that the model provides the students with opportunities for progress, without blocking the way back to the sources in which the understanding is grounded. The foregoing also means that the bar model can function on different levels of understanding, and that it can keep pace with the long-term learning process that students have to pass through. It is this enduring quality in particular that makes the bar model so powerful. Its flexible and general character expose the different appearances of rational numbers and their mutual relations; as a consequence of this, the students will get more of a grip on the underlying concept of rational number, which in turn combines with applying the model on a progressively higher level: from depicting a partwhole situation to local estimating and calculating to mathematical reasoning based on insights gained in (rational) number relations.

Just as it is not a case of *one* bar model, but of a chain of models that together form the conceptual model that incorporates the relevant aspects of the rational number concept, there also is not just one shift from model of to model for. In fact, there is *a series of continuous local shifts*, which implies that a model, which on a context-connected level symbolized an informal solution, in the end becomes a model for formal solutions on a more general level (Figure 16).

Such a local shift occurs for example when the students realize that the way in which the occupation of the theater is symbolized can also be used to express that 25% of the flowers are red. This shift in *context* is often the first step that gives a model a more general character. Another local shift concerns a shift in *(sub-)* domains, which opens the relationship between different (sub-) domains. This transition demands that the students understand that the same bar can be used for percentages as well as for fractions. Although the relation between these rational numbers based on well-known familiar fractions and percentages is an important cornerstone of the program, this shift occurs only when the children start making conscious use of it. Yet another local shift occurs when the different ways in which the model can function – and students can use it – are connected: what at first was just depicting is used later to estimate a percentage, or to calculate back from the new reduced price to the original price. This shift in *function* in the long run leads to the students being able to make flexible use of the model and manipulate it. At that point they have in effect reached the general, formal level of understanding.

Although a certain degree of ordering can be found in the different types of local shifts – the shifts in context, for instance, will often happen first – these must not be seen as being strictly sequential. Within the learning process the different local shifts are closely linked. Together they form the building blocks on the basis of which the rise in level of understanding is achieved.

Going through the different steps of the trajectory shown in this article, it seems that we have found a good scenario for teaching students percentage. This was further confirmed in the trial lessons through the experience that the perspectives of the developers, the teachers and the students appeared to coincide most of the time. Teachers could easily identify with the proposed trajectory. It was recognizable for them before they had carried it out themselves. In itself this inherent ability to convince is already telling. Even more important however is the experience that the students came up with solutions that were similar to the ones that were forecast in the trajectory. This experience truly gave the feeling of being able to accomplish what Streefland (1985b, p. 285) called:

> to foresee where and how one can anticipate that which is just coming into view in the distance.

However, these experiences must not result in concluding that this bar model based trajectory is the final answer to the question of how students can best learn percentages. It is just one answer. The trajectory depicted in this article should therefore not be seen as a fixed recipe, nor as a funnel in which the students have very few options to escape into finding another way of gaining certain insights, but as a *model for* teaching and learning percentage in which the didactical use of models plays a key role.

NOTES

1. This article is an adapted version of Van den Heuvel-Panhuizen (1995).
2. IOWO stands for *Instituut Ontwikkeling Wiskunde Onderwijs* (Institute for Development of Mathematics Education).
3. CRMSE is the predecessor of the National Center for the Improvement of Student Learning and Achievement in Mathematics and Science (NCISLA) at the University of Wisconsin-Madison.
4. Streefland (1985a, p. 63) put it in Dutch as follows: "In de mentale omslag van nabeeld tot voorbeeld worden bewustwording en niveauverhoging in het leerproces manifest."
5. The 'seating arrangements' (or 'table arrangements') refer to the way the children are seated in the pizza restaurant. The seating arrangement tells how many pizza are on the table and how many children are seated at that table.
6. This is the English translation of: "*Modellen van* iets zijn *nabeelden* van een stuk gegeven werkelijkheid; *modellen voor* iets zijn *voorbeelden* voor een te scheppen stuk werkelijkheid."
7. The draft version of *Per Sense* was developed by Marja van den Heuvel-Panhuizen and Leen Streefland. This took place from 1991 to 1993. The draft version of *Fraction Times* was developed by Keijzer, Van Galen and Gravemeijer. *More or Less* was designed in draft by Keijzer, Van den Heuvel-Panhuizen and Wijers.
8. The draft version of this unit was called *Travail Times*.
9. This elastic strip was an idea of Abels (1991).

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