SUPPORTING STUDENTS' QUANTITATIVE & COVARIATIONAL REASONING: DESIGNING & IMPLEMENTING TASKS LINKING DYNAMIC ANIMATIONS AND GRAPHS

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Researchers have incorporated dynamic environments involving multiple linked representations of covarying quantities when investigating middle and high school students' reasoning about change (e.g., Kaput & Schorr, 2008; Saldanha & Thompson, 1998; Stroup, 2002). A central conjecture underlying research related to one such environment, SimCalc, implemented extensively with urban students, involves the potential of representational structures to impact students' investigation of the important mathematical ideas of change and variation (e.g., Kaput & Schorr, 2008). Further, researchers using SimCalc environments with urban students found that students demonstrated both positive affect and powerful reasoning as they investigated change and variation (Schorr & Goldin, 2008). Using purposefully designed mathematical tasks this research intends to support students' drawing on their informal reasoning to mathematize (Freudenthal, 1973) situations involving covarying quantities. This study investigated the following question: How do students coordinate linked dynamic graphical and pictorial representations of covarying quantities?

Background

Central to our research is the investigation of students' reasoning about quantities that are covarying, or changing in relationship to each other (Carlson, Jacobs, Coe, Larson & Hsu, 2002). When indicating that students are reasoning about quantity, we mean reasoning about something as if it were a measurable attribute of an object (e.g., Thompson, 1994). Reasoning about an attribute of an object (e.g., area) as if it were measurable may or may not entail the act of measuring an attribute of an object (e.g., determining an amount of area). Further, we distinguish the act of evaluating numerical amounts (e.g., using a formula to determine amounts of area) from reasoning quantitatively (e.g., Thompson, 1994). Although evaluation may result from making sense of an attribute (e.g., area) as a quantity, when evaluating amounts a student may or may not be considering that attribute to be a quantity (e.g., a student could use the formula A=l*w with or without reasoning about area as measuring an amount of flat surface being covered). While numerical operations, such as the calculation of an amount of area using a formula, could be used to evaluate a quantity (Thompson, 1994) it is the conception of measurability—not evaluation—that is the essence of quantitative reasoning.

How a student conceives change in quantities can affect how that student may make sense of relationships represented by a graph (e.g., Johnson, 2012; Saldanha & Thompson, 1998). It is known that students can interpret graphs iconically such that graphs could have figurative similarities to a situation they are intended to represent (e.g., Clement, 1989; Leinhardt, Zaslavsky & Stein, 1990). For example, a student might interpret a graph representing a relationship between volume and height of liquid in a filling bottle as being shaped like the bottle itself. In contrast to an iconic interpretation, a covariational interpretation of a graph would involve considering a graph as representing a relationship between quantities that can change together (e.g., Carlson et al., 2002). Covariational interpretations of graphs can take different forms, involving chunky and smooth images of change (Castillo-Garsow, 2010; 2012; Castillo-Garsow, Johnson, & Moore, in press). For example, a chunky image of change would involve envisioning completed amounts of change (e.g., area changed by *a* units and height changed by *b* units), and a smooth image of change would involve envisioning continuing change (e.g., area is changing while height is changing). While dynamic environments linking animations with graphs might have potential to support students' interpreting and forming relationships between covarying quantities, students working in such environments seem likely to engage in a wide range of reasoning.

Dynamic computer environments that include multiple linked representations of covarying quantities can simulate change as continually occurring (Kaput & Schorr, 2008). Students working with dynamic computer environments, including computer graphing software, have demonstrated smooth images of change (e.g., Castillo-Garsow, 2010; Johnson, 2012; Stroup, 2002). An essential aspect of the SimCalc dynamic computer environments is the linking of motion simulations to dynamic graphical representations of changing quantities (e.g, Kaput & Schorr, 2008). Drawing on the work of the 14th century scholar Nicole Oresme, Kaput (1994) argued that images and representations of motion were critical to the development of the mathematics of change. As such, incorporating representations of motion (e.g., motion simulations) linked to graphical representations of changing quantities seems to be a crucial aspect of a dynamic computer environment designed to support students' reasoning about change in covarying quantities.

Dynamically Linked Graphs & Filling Area Animations

Drawing on the well-known bottle problem (Swan & the Shell Centre Team, 1999) and Thompson, Byerly, and Hatfield's (2013) version of the bottle problem that dynamically links a pictorial representation of a filling bottle with a graph relating the changing quantities of volume and height, Johnson used Geometer's Sketchpad Software (Jackiw, 2009) to develop dynamic environments to be used with middle school prealgebra students. Dynamic environments linked pictorial representations of an adjustable rectangle and a right triangle "filling with area" with a graph relating the changing quantities of volume and height (see Fig. 1, Fig. 2). For the filling rectangle environment, students could click and drag point F to vary the length of the base of the rectangle (See Fig. 1) For the Filling Triangle Environment, the length of the base, AB, was fixed (See Fig 2). To vary the area and height of the rectangle or triangle, students could click and drag on points H or D, respectively (see Fig. 1, Fig. 2) or press the action button Animate Point. If a student pressed animate point when EH (or AD) was greater than zero, the area would "fill" until point H (or D) reached the maximum height of the rectangle, then reset EH (or AD) to zero and begin filling area again until the original length of EH (or AD) was reached. Associated with each environment was a mathematical task[1] developed by Johnson. Results reported in this paper come from students' work on the filling triangle task.

[1] By mathematical task we mean a purposefully designed problem intended for a particular audience (Sierpinska, 2004).

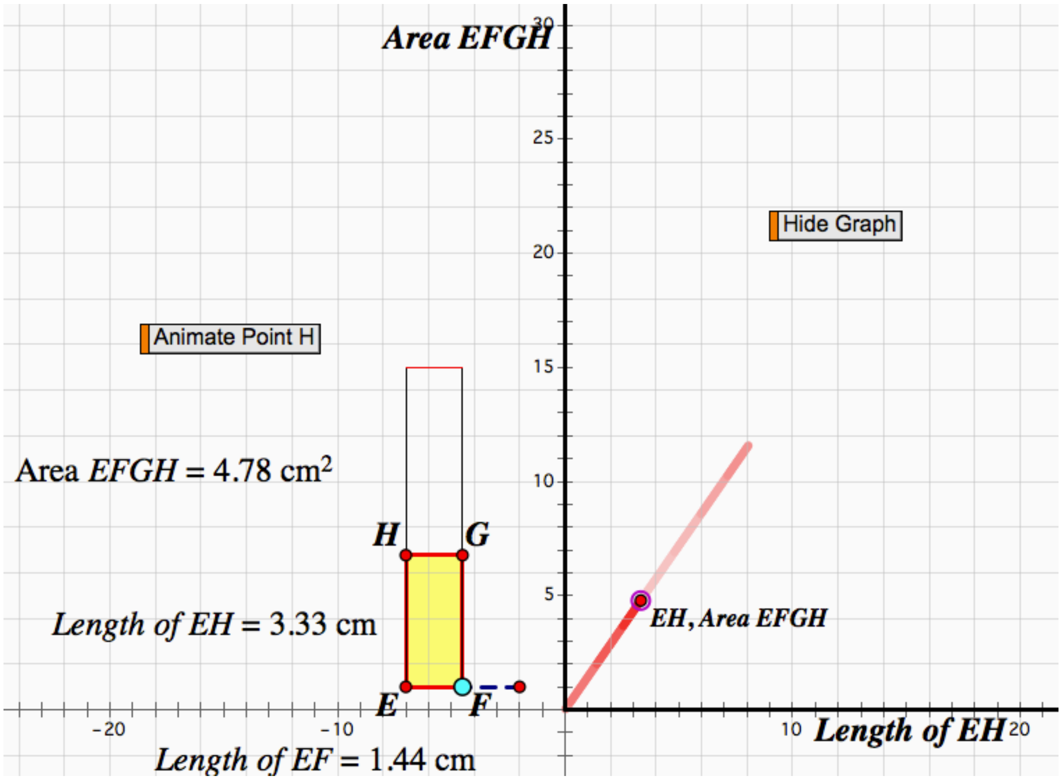

Figure 1: Filling Rectangle Environment

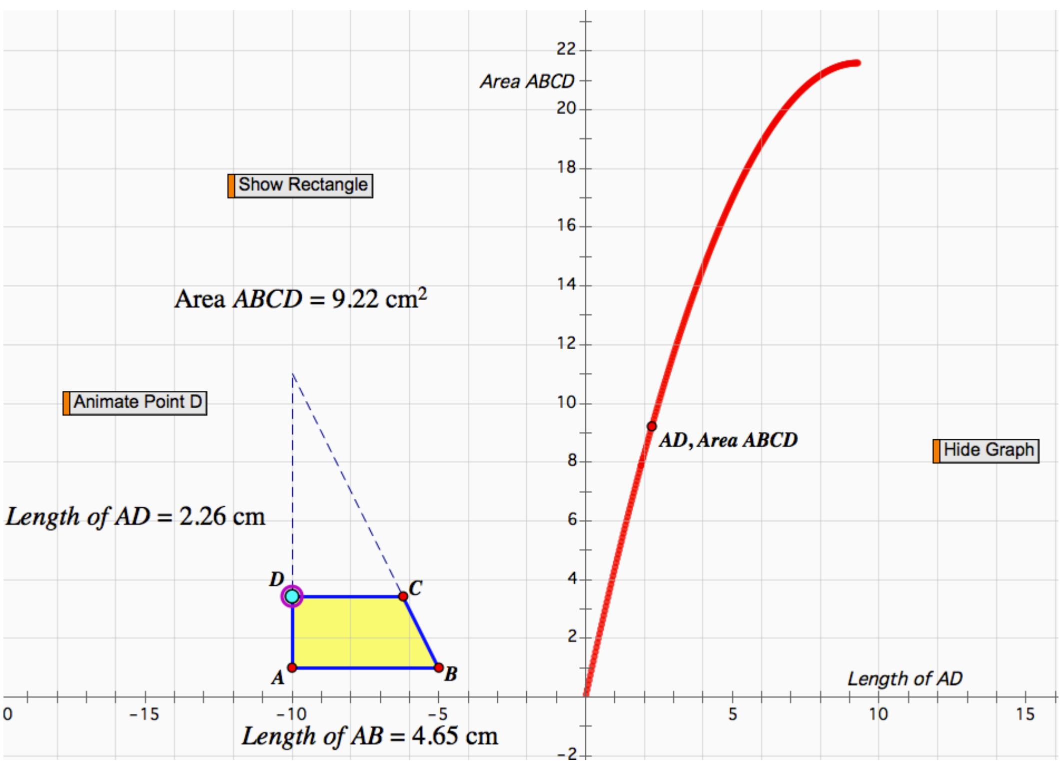

Figure 2: Filling Triangle Environment

The Filling Triangle task adapts the bottle problem in two ways: (1) Using a dynamic computer environment linking a dynamic pictorial representation with a dynamic graphical representation, and (2) Relating changing quantities of area and height, rather than volume and height. Anticipating that seventh grade students might have limited conceptions of volume, Johnson adapted the context of the bottle problem from a filling bottle to a two-dimensional shape being filled with area. In making this change, the two-dimensional pictorial representation would represent a two dimensional quantity rather than a three dimensional quantity. The Filling Triangle task involved four main prompts, shown in Table 1. Students responded to the first three prompts prior to running an animation linking the dynamically changing shaded region with the dynamically sketching graph.

1. Press *Animate Point* to run the animation of the filling triangle. What changes and what stays the same?
2. How does area change as the height increases?
3. Imagine you created a graph relating the side length of AD and the filling area ABCD. What would the graph be like?
4. Press *Animate Point* to sketch the graph. Was it what you expected? Why do you think it looks that way?

Table 1: Prompts included in the Filling Triangle task

Method

This research used design experiment methodology (Cobb et al., 2003) to investigate students' mathematical reasoning in classroom and small group settings involving tasks hypothesized to support students' reasoning about covarying quantities. Johnson served as PI for this study, conducted in May 2012 in an urban middle school serving a student population with high percentages of students receiving free and reduced lunch (90%) and students identified as English Language Learners (45%). Johnson led 6 days of whole class instruction with 4 sections of 7^{th} grade students, then conducted follow up task-based interviews (Goldin, 2000) with 7 pairs of students, selecting at least one pair of students from each section. Students were purposefully selected to participate in interviews, based on their willingness to participate in whole class mathematical discussions that involved students sharing their mathematical thinking. During the interview, student pairs first worked in the Filling Rectangle Environment (see Fig. 1), a follow-up to what students had done during classroom instruction, and then with the Filling Triangle Environment (see Fig. 2), an extension to what students had done during classroom instruction.

The task-based interviews were designed for the purpose of gaining further insight into students' reasoning. Essential to the design were carefully constructed prompts requiring students to predict features of a graph that had not yet been sketched and to interpret a given graph (static or dynamic) in terms of the quantities involved (area and length). When predicting graphs, we provided students with Wikki Stix© (bendable wax coated sticks of yarn) to support students' consideration of graphs in ways other than a collection of points. During the interviews, Johnson encouraged students to use familiar language. Sometimes students used one word (e.g., square) when they seemed to mean another word (e.g., rectangle). If this happened, Johnson asked for clarification only if necessary to clarify students' reasoning.

Data analysis involved multiple passes. Annotated transcripts, video recordings, and students' written work served as sources of data. In the first pass we identified instances of data when students predicted or interpreted features of a graph linked to an animation. In the second pass we engaged in open coding (Corbin & Strauss, 2008), relating portions of data based on how students drew on quantities represented by the animations to account for different features of a graph. In the third pass we engaged in the constant comparative method (Corbin & Strauss, 2008), working from students' attention to changing quantities to develop explanations of students' reasoning that could account for students' prediction or interpretation of a graph.

Results

We report four interview excerpts, each from a different pair of students working in the Filling Triangle Environment, representing the range of reasoning demonstrated by all 7 pairs of students. Excerpts were typical of interactions that occurred during the interview with each pair of students. Prior to these excerpts, students worked in the Filling Rectangle Environment. Accompanying the Filling Rectangle Environment were three static line graphs sketched on the same plane, each representing a relationship between the area of rectangle EFGH and the side length of EH for three different lengths of EF. Students were prompted to determine the length of EF for a rectangle represented by each graph and to approximate the length of EF for a graph described in relationship to the given graphs. All students were able to make comparisons between lines having different steepness, determining that the steeper a graph representing the area of a rectangle and the height of a rectangle, the larger the base of the rectangle.

When working in the Filling Triangle Environment each pair of students was prompted to predict then interpret features of a graph. Prior to predicting, students were asked to explain what the graph was representing. At least one student in each pair explicitly indicated that a graph would represent a relationship between the length of AD and area of ABCD. The first two excerpts report students appealing to a procedure for calculating area or to a shape of a geometric object to predict or interpret features of a graph relating the area of ABCD and the length of side AD. The last two excerpts report students appealing to how area might change to predict and interpret features of a graph relating the area of ABCD and the length of side AD. Student names are pseudonyms. In each excerpt, when we use "graph" we mean a graph relating the area of ABCD and the length of side AD, see Fig. 2.

Using the shape of a geometric object to predict features of a graph: Navarro & Daria

Prior to this excerpt, Navarro had expressed hesitation about whether or not a graph would be linear. He compared the Filling Triangle to the Filling Rectangle, indicating that for the Filling Rectangle both vertical sides of the rectangle were changing (see Fig. 1), but for the Filling Triangle only the vertical side was changing, because the length of BC (see Fig. 2) would not be multiplied by the length of AB to determine area. As Navarro started to contrast the Filling Triangle with the Filling Rectangle, Daria quietly said "only one," seeming to concur with Navarro regarding how only one side would be changing. The excerpt that follows occurred just after this interaction, with Johnson (Researcher) prompting Navarro and Daria to predict features of a graph.

Researcher: What do you expect that graph would be like?

Navarro: It would go like up. *[Positions forearm at approximately a 20 degree angle with the horizontal, with elbow in the air, then moves forearm to approximately a 60 degree angle with the horizontal, while moving elbow to the right.]* I feel like it—

Researcher: It would go like up. So Daria, this is what Navarro thinks, he thinks it would go like up, *[Positions forearm to mimic Navarro's.]* what do you think?

Navarro: *[Said while Daria is working.]* Yeah, I changed my mind, I think it will be- *[Raises forearm back up at approximately a 60 degree angle with the horizontal.]*

Daria: *[Beginning near the origin, moves the tip of her pencil along the grid on the handout in a manner seeming to indicate a line with a positive slope.]* It would**.**

Navarro: Yeah, cause you're not adding anything really. You're not adding part of the- *[points to DC on the filling triangle]*, that side, you're just, you keep multiplying the exact same two sides, but, cause, you're not changing the length of AB, you're just changing your length of AD.

Researcher: And Daria, why do you think it would be linear?

Daria: Mm. Because it's not skipping around. And um, well if you make it go by one, *[Moves pencil along length axis, touching what appear to be consecutive hash marks, lifting her pencil after touching each hash mark.]* and you mul—, I just don't get it because, how do you get from this *[Uses pencil to point to Length of AD]* to that *[moves pencil to Area of ABCD]*?

Researcher: Ah, like how do you actually calculate the numbers?

Daria: Yeah.

Researcher: And yeah, that's, we didn't show you that.

Daria: Yeah, that's, I need that in order to— *[Voice trails off]*

Although Navarro initially expressed uncertainty about whether or not a graph would be linear, his gestures associated with his predictions of a graph seemed to represent a linear graph. Further, right after this excerpt, before viewing the dynamically sketching graph, Navarro remarked "I think that it'll be linear," appealing to sides being multiplied to explain why he changed his mind. Noteworthy is both Navarro's and Daria's drawing on how area would be calculated to support their predictions. By expressing a need for procedure for calculating the area of ABCD given the length of AD, it seems that Daria needed to work from calculations to make sense of a graph relating side length and area. Even when particular values were included, it seemed to be insufficient for Daria because no procedure linking the values was given.

Using the shape of a geometric object to interpret features of a graph: Blanca & Crista

Prior to this excerpt, both Blanca and Crista had predicted that the graph would be linear. Crista predicted that the line would be not too steep because "it's not a full rectangle." Blanca remarked that one could "ignore the fact that it's a triangle" and make sense of the situation in a way similar to the filling rectangle. This excerpt begins with Johnson (Researcher) prompting Blanca and Crista to use the computer to sketch a graph.

Researcher: So, if you press show graph and then you animate the point, you'll be able to see the graph that the computer constructs.

Blanca: Oh, um.

Crista: Never mind. But look it's curving.

Blanca: Oh, but it curves *[Said while Crista is talking.]*

Researcher: Why does that make sense?

Blanca: It's not linear.

Crista: Because, okay, so let's, since it's half of a rectangle. Rectangles are almost all the time linear. It would like, so since it's half of a rectangle, it would be linear, no, non-linear. So that's, that could mean that the whole rectangle would be linear, because, you still need the other half.

Researcher: What do you think Blanca?

Blanca: Um, I, of why it's not linear? Okay, well I don't think it's linear, no, because, it's like, I think it's probably for, cause it's not a triangle the whole time. Cause like the rectangle, it was a rectangle the whole time. But this is a trapezoid and then it eventually grows into being a triangle. So it would be a different story if it was a triangle the whole time.

Crista accounted for the curvature of the graph in terms of the nature of the shape such that rectangles produce linear graphs, but triangles produce a different kind of graph. Blanca provided additional information in terms of the nature of the shape of the shaded region—for a triangle the shape will be a trapezoid rather than a triangle. As such, both Blanca and Crista used the nature of the shape of the animation to interpret features of the graph.

Predicting features of a graph: Simon & Tomas

In this excerpt both Simon and Tomas used "constant rate" to explain their predictions. When working with the Filling Rectangle environment, Simon described the filling area of the rectangle as having a constant rate of change, clarifying that "the area increases continually by the same amount." With the Filling Triangle Environment, both Simon and Tomas used "constant rate" to describe how the numerical amounts of area (shown in the dynamic computer environment, see Fig. 2) were changing. The excerpt begins with Johnson (Researcher) asking Simon and Tomas to predict features of a graph.

Researcher: What do you think a graph would be like?

Simon: I think since it's not changing by the same amount every time that it would be not linear. It would be curving because it's not increasing by a constant rate of change.

Researcher: Ah, do you think it would be curving too? *[Said to Tomas.]*

Tomas: Yes.

Researcher: What kind of curve would it be like? *[Beginning near the origin and moving from left to right, Tomas presses a Wikki stick onto the coordinate plane, bending the stick to represent a curve increasing to a maximum, then decreasing.]*

Tomas: It's going to be like this. *[Tomas runs his finger along the Wikki stick, moving from left to right.]*

Researcher: Okay. Why like that?

Tomas: Cause first it's going at a constant rate. And then once it gets to like twenty-one, or twenty—it starts getting slow and then it's not going at a constant rate anymore.

After this excerpt Simon asked Johnson (Researcher) if he might be able to use a value of 2 for AB and determine some amounts of area. Beginning with a length of 1 for AD and increasing by increments of 1 until AD was 3, Simon determined amounts of area by performing the operation AB*AD/2. He pressed his Wikki stick onto the graph incrementally, moving it as he determined each value. In contrast Tomas created graphs without making any calculations. Instead Tomas appealed to the "constant" then "slow" increase in area. Without prompting by Johnson (Researcher) he revised his graph, which originally looked like a parabola with a global maximum, to create a monotonic graph increasing with different magnitudes.

Interpreting features of a graph: Tien & Jorge

Prior to this excerpt Jorge had predicted that a graph relating area and side length for a triangle would be "mostly on the linear side". When asked what it would mean for a graph to be mostly linear, Tien explained that the graph would be linear from the base of the triangle up to a certain point (when the height of AD was approximately 2/3 its maximum height). Tien remarked that after that certain point it would "go like that more," sketching a graph in the air that appeared to be piecewise linear, with the first portion having a larger positive slope than the second portion. When prompted Jorge added that the graph might "go up and then go down a little bit and probably travel back up." After viewing the dynamic graph shown in Fig. 2, both

Tien and Jorge expressed pleasure in the fact that the graph curved, with Tien saying a quiet yet excited “yes” as she watched the graph. At this point, Johnson (Researcher) prompted students to interpret the graph in terms of area and side length.

Researcher: Can you use the, can you use the area and the side length to explain why it might make sense that it's curved?

Jorge: So, the length of A and B, it like stays the same throughout the whole way, and then the length of AD changes as it goes up, and then the area would eventually turn out to be like, maybe get smaller.

Researcher: Now, before, we said the area increases the whole time, right, like as you increase it? So when you said it gets smaller, do you mean that you're losing area?

Tien: No, like you're getting a sm—, like a smaller, well, like a smaller amount of area added, so—

Researcher: Can you show me that? Like can you show me that either with the animate point, or like, a smaller amount of area added-

Tien: So like, from right here, *[Clicks and drags point D. Moves point near the tip of the triangle.]* like it's getting a little bit of more area *[While talking, slowly drags point D, moving point D closer to the tip of the triangle.]* than how it was like from down here, *[Points near the base of the triangle.]* so technically it's like going at a slower rate, than how it is, so probably like, because only the decimal's moving, *[Points to Area ABCD label on vertical axis.]* like the decimal's probably in the, only smallest point, so it’d probably be like it'd be going, be going um, a little smaller than, well slower than how it is, so it's probably not going straight, because it's getting an even amount of time to like grow, so technically the timing will be a lot more slower once it gets to the top.

Jorge: So like, to agree with Tien, there’s like, so if you see down here the graph goes by really fast.

Researcher: Okay, could you show me that, like if the graph goes by really fast, you can either press animate point, or you can—

Jorge: So when you animate it, it went up really fast, and then like as soon as it got to that one point, it started to get slower and slower, and added on a little bit slower.

When explaining why the graph curved Tien appealed to amounts of area being added. When prompted to show what “a smaller amount of area” would be, Tien dragged point D to vary the area of the triangle, simultaneously referring to the movement of the decimal points indicating the amounts of area. While Tien attended to the area and its related numerical amounts, Jorge focused on the speed at which the graph sketched, using the variable speed of the graph to explain why the area was increasing at a magnitude that was decreasing.

Discussion

Students reasoning about area as a result of a calculation made comparisons between the shape of an animated object and the shape of a graph and made sense of variable increase as if it were constant. When making sense of a relationship between the area of ABCD and changing length of AD, students envisioned area as something that could vary. If students execute a procedure to determine an amount of area prior to making sense of area, then consideration of how area might change would occur after amounts of area have been determined (e.g., Navarro & Daria). Students who reasoned about area by interpreting amounts of area determined by a procedure made sense of an unfamiliar graph by connecting shapes of objects to shapes of graphs such that rectangles have one type of graph and triangles have another (e.g., Blanca and Crista).

Both pairs of students made predictions that the graph relating the filling area of triangle ABCD and side length AD would be linear. Perceiving a linear relationship in this situation could be tied to observing that the height of the triangle in the animation was increasing steadily. This research suggests that iconic interpretations can extend to dynamic graphs such that dynamic graphs are pictures in motion.

Unlike students who made sense of dynamic graphs as pictures in motion, students reasoning about area as a measurable attribute of a rectangle or triangle could consider how area might increase at different magnitudes when interpreting and/or predicting features of a graph relating area and side length. Students' envisioning of changing area took different forms: considering how a set of pairs of area and side length might suggest a relationship that would be a nonconstant increase (e.g., Simon), considering sections when area would be increasing at different magnitudes (e.g., Tien), and considering how area could increase at different magnitudes as area continually changed (e.g., Tomas). The different ways in which students envisioned area to be changing provide examples of chunky images of change (e.g., Simon) and smooth images of change (e.g., Tomas). While being able to imagine continuing change seems critical for students to attend to variation in the intensity of an increase (Johnson, 2012), investigating incremental change (e.g., Simon) seems essential for coordinating amounts of change in covarying quantities.

While students considered change in covarying quantities when working in both the Filling Rectangle and the Filling Triangle Environments, it was students' work in the Filling Triangle Environment—involving a varying rate of change—that afforded us the opportunity to make these distinctions in students' reasoning. It was not until the use of a procedure to calculate area was problematized in the Filling Triangle Environment that distinctions emerged between how students were making sense of area as quantity and how students were considering area as increasing at different magnitudes. Consistent with other researchers (e.g., Stroup, 2002), our study suggests that situations involving a constant rate of change may not be sufficiently complex to identify distinctions between students' quantitative and/or covariational reasoning.

Concluding Remarks

Dynamic environments linking animations involving covarying quantities with graphs representing relationships between those covarying quantities seem useful for supporting students' interpretation of graphs as relationships between quantities. However, to interpret linked graphs from a covariation perspective, students need to consider animations as relationships between quantities, not just a picture in motion. Key elements of tasks used in this research were prompts requiring students to predict how change might continue. Such prompts provide opportunities for students to make sense of change as a quantity that can vary in intensity, not just as a result of a calculation. Students' emerging ways of reasoning could provide benchmarks indicating how students might develop in quantitative and covariational reasoning. Research investigating how students interpret and create relationships between changing quantities seems useful for developing trajectories of reasoning related to the difficult to learn concept, rate of change.

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