Virtual Environment Technology: Developing Students' Abilities to Apply Calculus

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This research utilizes Virtual Environment (VE) to connect students' calculus knowledge with a corresponding reality. The study explores how students, who had completed an AP calculus course, find the optimal path in a VE empirically and, after that, mathematically. The basis for data examination is Realistic Mathematics Education theory. Theoretical constructs of 'intuitive cognition' is also used. An interactive setting for the empirical optimal path-finding is programmed in the Second Life VE. The data recorded from the activities of five out of ten students, participated in the study were selected and analyzed. The results demonstrate that students constructed their models-of the situational problem on the basis of their VE activities. The results also show that new VE empirical knowledge prevails over intuitions.

Keywords: Virtual Environment; real-life problems; Realistic Mathematics Education; mathematizing; intuition

1. Introduction

The contemporary application of calculus to real-life processes includes numerous science/engineering fields such as fluid and air dynamics, mechanics of solids, theory of control of moving bodies, and a variety of biological and ecological processes. This in turn, requires from graduates strong ability to apply their calculus knowledge to reality. In the late 1980s the Calculus Reform movement began in the USA. The Calculus Consortium at Harvard (CCH) was funded by the National Science Foundation to redesign the curriculum with a view of making calculus more understandable, more applied, and more relevant for a wider range of students. One of the desired characteristics of calculus course was that students and instructors would find the applications real and compelling (Tall, Smith, & Piez, 2008). Consequently, many teachers and textbook writers have been working on the development of mathematical school tasks that resemble out-of-school situations. Nevertheless, the recent research shows that graduates do not know how to apply knowledge to many problems that arise outside the walls of school (Ilyenkov, 2009), and this is still a troubling problem with current education. A serious mismatch exists and is growing between the skills obtained at schools and the kind of understanding and abilities that are needed for success beyond school (Lesh & Zawojewski, 2007). The idea of including the 'out-of-school' world in mathematics education, implying that a focus be put on real-life applications, is not new and has been emphasized in education policy in many countries (Palm, 2009). Long ago, Freudenthal (1968) raised the problem of the lack of connection between mathematical classroom knowledge and real-life objects/situations. He pointed out that the problem is not the kind of mathematics being taught, but how it is taught. Almost 30 years after Freudenthal's first claim, Davis (1996) stated, "... with reference to school mathematics, the subject matter has come to be regarded as having little to do with the "real world" and as bearing an even more tenuous relationship to the lived experience of learners" (p.88).

A traditional way of description of the contextualized tasks containing out-of-school real life situations is a so called '*word problems*'. Word problems are firmly entrenched as a classroom tradition, particularly in North American schools (Gerofsky, 1996), and yet, there has been long lasting debate about the reasons for word problems' lack of effectiveness as a link between abstract mathematics and real-life phenomena. Gerofsky (2006) asserts that word problems are unable to be faithful simulations of real-life tasks. She insightfully predicts that new approaches should appear based on new computer technologies.

There are many studies devoted to using different types of computer technologies in schools for the teaching and learning calculus. Extensive review of such studies is provided in (Tall et al., 2008). The various approaches using different technological tools for teaching and learning calculus are intended to improve understanding of the main calculus concepts, which was undoubtedly a very important goal. The purpose of this study is to utilize contemporary technologies to 'bring' the physical world into classrooms so that the students can apply their calculus knowledge to real-life problems. The particular goal is to utilize Virtual Environment (VE) technological tool in order to create a real-life situational problem in classrooms and let students explore it empirically and mathematically.

2. Virtual Environment: A Tool for Simulation Reality

The departure point for this study is the assumption that Virtual Environments (VEs) are the contemporary technological tool which can 'bring reality' into classrooms. VEs represent physical world situations with a high degree of fidelity. For example, immersion in the *Second Life* VE erases the difference between real and virtual worlds to the extent that users' psycho-physical behaviors in the VE are consistent with real life (Massara, Ancarani, Costabile, Moirano, & Ricotta, 2009). Many recent publications are devoted to the *Second Life* VE, to its popularity and application (e.g., Boulos, Lee, & Wheeler, 2007; Campbell, 2010; Massara et al., 2009; Messinger et al., 2009). *Second Life* (http://www.secondlife.com) is an accessible and easy to use VE; it has 3D computer graphics and high fidelity; it provides egocentric and allocentric perspectives. High fidelity, egocentric/allocentric perspectives, and the interactive nature of the environment create the perception of *presence*. Many “residents” of the *Second Life* VE are escaping from their everyday real lives into this synthetic world (Messinger, Stroulia, Lyons, Bone, Niu, Smirnov, & Perelgut, 2009). This, in turn, means that the VE synthetic world itself becomes a reality for VE users. Moreover, the term ‘Virtual Environment’ is also referred to and widely known as ‘Virtual Reality’ (VR), which reflects its essence of ‘reality’.

All described above characteristics of the *Second Life* VE determined our choice of utilising it for this study and also determined the choice of theoretical background for experimental design and data analysis.

3. Theoretical Framework

The background theory of this research is *Realistic Mathematics Education* (*RME)* instructional design theory, the main characteristic of which is the use of *realistic contexts* which should, in turn, should be suitable for *progressive mathematizing*. Treffers (1987) formulated the *progressive mathematizing* as a sequence of two types of mathematical activity – *horizontal mathematizing* and *vertical mathematizing*. *Horizontal* refers to transforming a nonmathematical problem field, related to a real world situation, into a mathematical problem. *Vertical mathematizing* is grounded in the horizontal and includes such activities as reasoning about abstracts and structures within the mathematical system itself. Another characteristic of RME is modeling principle which is expressed through construction of *models-of* a VE ‘real-life’ activity and shifting from a *model-of* to a *model-for* mathematical reasoning (Van den Heuvel-Panhuizen, 2003).

This study also utilises Fischbein’s theoretical construct of *tacit intuitive model* to analyse an influence of intuition on VE empirical activity. Fischbein (1989) describes the model as mental, intuitive, tacit, primitive, global, and meaningful interpretation of a phenomenon or a concept. A fundamental characteristic of a *tacit intuitive model* is its robustness and capacity to survive long after it no longer corresponds to formal knowledge.

4. Contextual Problem

A contextual problem for this research design is the optimal navigation when traversing two different media. This problem allows for developing informal solution strategies; it can be represented on the computer screen and explored by students empirically; it is experientially real for everybody and suitable for mathematizing. It can be potentially transferred into corresponding calculus task, available in almost every calculus textbook (Pennings, 2003). The detailed description of the calculus task (with two different media of water and land) is provided in (Pennings, 2003). The main idea of the task is to reach an object B, located in water, from position A, located on land close to the water's edge, and to find such a path that would minimize the time of travel from A to B (see Figure 1).

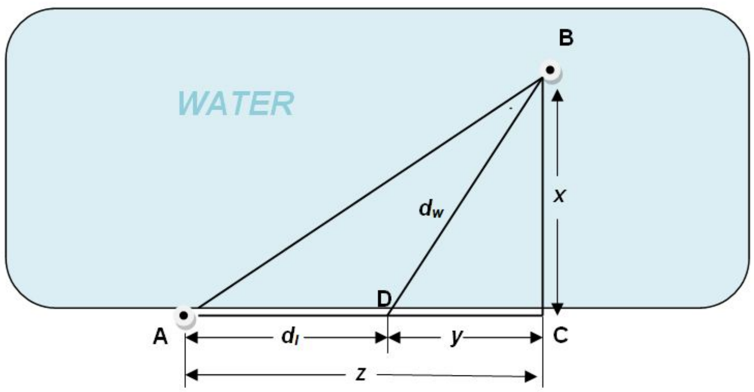


Figure 1. Possible Paths: From Location A on Land to Location B in Water.

Path AB is the shortest distance between A and B, but it also has the longest water distance between the points. Since the speed in water is slower than the speed on land, the choice could be to use the shortest water distance, which means sprinting down the beach to the point on shore closest to the 'water' platform, which is C, and then turning a right angle and moving to B. Finally, there is the option of using a portion of the land path, up to D, and then entering into the water at D and moving straight to the water platform.

Let z denote the distance between A and C; d_l denote the distance between A and D, that is, the distance traveled on land. Let $y = z - d_l$, and x represent the distance between B and C. Speed on land is s_l; speed in water is s_w. Then time spent for the trip is

$$T = \frac{z-y}{s_l} + \frac{\sqrt{x^2+y^2}}{s_w} \qquad (1)$$

The condition of minimal time:

$$\frac{dT}{dy} = T^{/} = 0 \;\Rightarrow\; T^{/} = \left(\frac{z-y}{s_l} + \frac{\sqrt{x^2+y^2}}{s_w}\right)^{/} = 0 \qquad (2)$$

Solving (2) for y we get:

$$y = \frac{\boldsymbol{x}}{\sqrt{\frac{s_l}{s_w}+1}\sqrt{\frac{s_l}{s_w}-1}} \qquad (3)$$

Analysis of the final solution (3) shows that the optimal path does not depend on z, as long as z is larger than y**.** There is no solution if s_l is smaller than s_w.

5. Methodology

5.1 *Second Life Task Design*

The simulated in the *Second Life* VE setting includes a pond, surrounded by bushes and trees (Figure 2). The environment was programmed so that walking/running speed on land was twice as fast as walking/running speed in water. There are two small round green platforms: one platform is located on land near the water's edge, another is located in the water (see Figure 2).

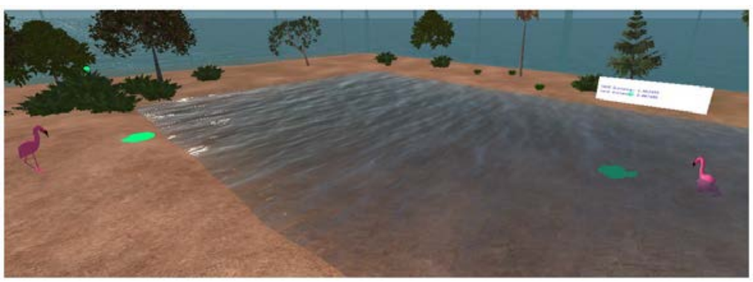


Figure 2. Simulation in the Second Life VE Setting

Note. One small green round platform is on land (left part of the figure), another is in the water (right part of the figure). Recorded data appears on a white banner (top right corner of the figure).

The VE task is to find the path between the platforms which would minimize the time of travel. The setting is programmed to record the total time and the distance traveled on land for each trip between the platforms. This information is indicated on white banners, one of which is shown in Figure 2. After each trip, the student must transfer the data from the banners (total time and distance traveled by land) into a specially designed guiding-reflecting journal, which is an integral methodological part of the research design.

5.2 *The Guiding-Reflecting Journals*

The goal of designing of guiding-reflecting journal was to provide students with the guidance which they may potentially need during completing the task. One of the basic principles of RME instructional design theory is a guided reinvention approach to teaching and learning. According to the theory, the teacher provides guidance, playing a 'proactive role' within the classroom setting. In this study, each student had to decide whether and to what extent s/he needed guidance, and this guidance was provided in the guiding-reflecting journal. It was expected that students might wish to develop their own *models-of* the situational problem without guidance. The journal contains blank space for such independent reasoning. The provided guidance corresponds to the optimal path finding calculus task described above. Therefore, students had a free choice: either to construct and develop their own *models-of* and *models-for* or to accept and develop the journal's model.

5.3 *Participants and the Stages of Designed Study*

Ten students from Vancouver's Templeton Secondary School, ranging in age from 17 to 18 years, 5 males and 5 females, participated in the study. They were at the end of an AP Calculus course and had completed such topics as application and computation of derivatives. They were all informed about the goal of the research and that the experiments would be conducted in the school's Teachers' room, outside of regular calculus class time and that each session would last 60-90 minutes.

The study contained four stages. The first stage of the experimental design is called an 'exploration trial' as in (Mueller, Jackson, & Skelton, 2008). This first stage allowed for students' free activities in the VE before they got the journal with instructions for the VE task. The exploration trial was of unlimited duration, lasting until the student felt comfortable in the environment and announced that s/he was ready to start the next stage. This first stage allowed students to explore the pond and to feel the speed difference on land and in water. Altogether, the goal of the exploration trial was to let students get the feeling of 'being' in the environment. At the beginning of this second stage, students received the guiding-reflecting journals with instructions. Their optimal navigation investigation in the VE was accompanied by working with the journal. The third stage of the designed study was mathematizing the VE activity which entailed journal work exclusively. The final, fourth, stage of the experimental design involved completing the journal's questionnaire.

The data is drawn from 3 sources: video recordings of students' mathematizing in the guiding-reflecting journals, screen-capture of their VE activities, and the guiding-reflecting journals.

6. Results and Analysis

Data recorded from the activities of five out of the ten students who participated in the study were chosen for detailed analysis. These five students were selected because they performed five different ways of mathematizing, which in turn allowed exploring the differences. The first volunteer, Kenneth, was the most motivated and interested in calculus. The analysis of his activity is presented in section 6.1. The activity of the next participant, named Jason, is analysed in section 6.2. The analyses of the activities of the other three students, Nick, Kate, and Ann, are shown in sections 6.3, 6.4, and 6.5, respectively.

6.1 *Kenneth*

Kenneth spent 3.45 minutes exploring the environment and had an opportunity to feel the speed difference. He also was informed by the journal instructions that the speed on land is two times faster than the speed in water. Nevertheless, Kenneth's first trip (he preferred to begin with the return trip) was a straight line between the platforms which is the longest water path as shown in Figure 3, a).

Figure 3. ***Kenneth's Two Initial Strategies: a) First Trip Strategy; b) Second Strategy***

It is self-evident, certain, and intrinsic, that the shortest distance between two points is a straight line (Fischbein, 1999). Kenneth chose the straight line as a shortest distance, having in mind an intuitive model that the shortest distance would give him the shortest time. Kenneth's tacit intuitive model prevailed over his knowledge, due to its robustness (Fischbein, 1989). After completing two trips using the strategy of a straight line between the platforms Kenneth planned and completed two trips of minimizing the water distance of the types shown in Figure 3 (b) above. Kenneth's diagram in Figure 4 represents a right triangle, reflecting the first two strategies of maximal and minimal water distances.

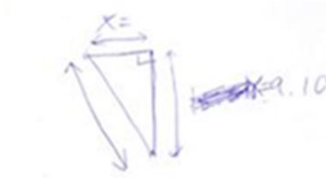


Figure 4. ***Kenneth's Diagram Reflecting the First Two Strategies***

Treffers (1987) claimed that mathematizing is paved via model formation, schematizing, and symbolizing. Therefore, according to this, Kenneth's diagram in Figure 4 above indicates the beginning of the mathematizing process. Figure 5 represents Kenneth's plan for a new trip, which was supposed to be between the two first paths of maximum and minimum water distances.

Figure 5. ***Fragment of Kenneth's Activity Transcript: Trip Planning***

After completing the Figure 5 strategy, Kenneth found out that it gave him a smaller time value than the previous attempts. Altogether, Kenneth used three strategies and completed two trips (forward and return) using each of them. After testing all of them, Kenneth decided that he had enough empirical data, articulating, "I really think I tried all the paths..." . This particular moment is a shift from the empirical physical world to the mathematical world. Kenneth starts his mathematical activity by drawing the diagram incorporating all three strategies which he used in VE (see Figure 6).

Figure 6. ***Diagram: Shift from the Empirical World to the World of Mathematical Abstracts***

The diagram in Figure 6 is a first step to mathematical justification of the collected empirical information. From the perspective of RME, the diagram in Figure 6 is a *model-of*

Kenneth’s informal activity in the VE and also a transition into a *model-for* formal mathematical reasoning.

Framing the process of Kenneth’s mathematical activity in terms of multiple layers of horizontal and vertical mathematizing (Rasmussen, Zandieh, King, & Terro, 2005), it should be stressed that Kenneth started to mathematize the problem while collecting empirical data (e.g., diagram in Figure 4 and a diagram of planning a new trip shown in Figure 5). The existing descriptors of mathematizing in literature do not accurately capture this part of Kenneth’s activity. Therefore, for the process of developing empirical knowledge from deliberate planning and execution of trip strategies, we consider it reasonable to introduce the new term of *empirical mathematizing*. Empirical mathematizing refers to the systematic planning and execution of an object-sensory activity for collecting and organizing empirical data within the problem situation for further “transforming a problem field into a mathematical problem” (Treffers, 1987, p.247).

Kenneth had a natural habit of articulating his mathematical reasoning out loud. Figure 7 shows Kenneth’s first formula and way of thinking (aloud) while writing it.

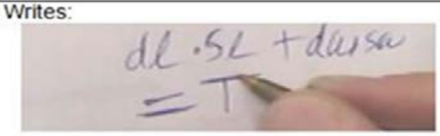
24:14	**24**	Writes: 	(0.6) OK so…a-am (0.4) so distance on water times the speed of water plus the distance on land times the speed on land is equal to the time spent for the trip =

Figure 7. Kenneth’s First Formula

Kenneth writes an abstract formula, but reasons in ‘real-life’ terms which in turn agrees with Freudenthal’s (1991) viewpoint that the distinction between horizontal and vertical activity is not rigid and structural. Keeping in mind the physical meaning of variables, Kenneth, nevertheless, made a mistake in this relationship between distance, speed, and time, which has a strong physical sense and relates to science.

Kenneth’s shift to vertical mathematizing, fully detached from horizontal, occurred at the moment when he articulated, “some trick in geometry” and then draws a right triangle with the form and symbols, corresponding to the traditional ‘geometrical’ representation which appears in every textbook (see Figure 8).

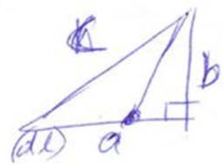
26:49	**29**	Draws a new diagram: 	=(0. 11) so
27:09	**30**	Writes : $\sqrt{(a-dl)^2+b^2}$	<u>Oh</u>… <u>I get it now</u>! (0.3) *a* minus (0.15)

Figure 8. Kenneth's Shift to Vertical Mathematizing, Detached from Horizontal

This new and fully abstract geometrical representation triggered finding the fully abstract formula shown in line (30) accompanied with the exclamation: "Oh… I get it now!" This traditional right triangle "provides the means to detach a concept from its concrete embodiment" (Herscovics, 1996, p. 358). The activities shown in Figures 9 refer to vertical mathematizing.

27:38	**32**	Finishes formula for d_w from <line 27>: $dw = \sqrt{(a-dl)^2+b^2}$	(0.2) am...(while) *a* and *b* are like the max things I am willing to go…I think…so then (0.3) you find the derivative of d_w with respect to…am...m...so..
28:09	**33**	Writes: $\frac{dw}{ddl} = \frac{\pm(a-dl)}{2\sqrt{(a-dl)^2+b^2}}$	(0.8) am..m...(0.7) derivative…OK…so…just using the chain rule …am..m..m (0.12) two times *a* minus d_l times (negatives) one (isn't a chain rule)… and the twos cancel...so (0.3)=

Figure 9. Fragment of Kenneth's Vertical Activity Transcript

An interesting moment happened in (Figure 10) when Kenneth obtained a "plus or minus" result, which obviously is a mathematical abstraction. He noticed, "then land equals…like plus or minus" (Figure 10).

33:20	**41**	Writes: $\pm b = \pm a \pm dl$	= and then you square root so you have to do plus or minuses...a plus or minus *dl* ...and then land equals... like this plus or minus

Figure 10. Kenneth's Contradictory Results

This 'plus or minus result' meant for Kenneth a contradiction with physical reality. The contradiction with physical reality made Kenneth believe that he had made a mistake in his calculations. He starts the calculations from the very beginning and makes two mistakes in the same formula connecting time, distance and speed. As it was mentioned above, this formula relates mainly to science, but not to calculus and demonstrates a lack of connection in Kenneth mind between mathematics and science. Kenneth was concentrated on mathematics. He applied calculus to the problem, namely, he applied Maxima and Minima theory; he also applied the chain rule in his calculations (see Figure 9). Nevertheless, the mistakes in the initial formula did not allow Kenneth to find a 'physically real', correct solution to the problem. From a RME perspective, this is evidence that "vertical mathematizing consists of those activities that are grounded in and build on horizontal activities" (Rasmussen et al., 2005, p. 54). After a new attempt to re-compute everything, Kenneth was stopped because of time restrictions on the session.

6.2 *Jason*

Jason spent only 30 seconds on the exploration trial, so, obviously he didn't have an opportunity to feel the difference between speeds in water and on land. He chose his first trip strategy (see Figure 11) under the influence of the information that speed on land was faster than speed in water. Jason received this information before his first VE trip from the journal.

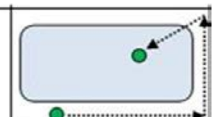
02:17	1	Completes the forward trip:	
02:47	2	Transfers data about <line 1> trip into the journal table of Attempt 1, < T=27.952750 s>; writes reflections about the trip: land is faster	

Figure 11. ***Fragment from Jason's Activity Transcript: First Trip***

All the other Jason's strategies were strategies of either minimizing or maximizing the distance traveled by water, except one, which was between them and, ideally, should give Jason the best time. Unfortunately and unexpectedly, the computer program did not show this best time on the banner. As a result, Jason's best time, was one of the in the shortest water distance trips. The repeated extreme strategies which Jason used during his empirical mathematizing were integrated in the diagram shown in Figure 12.

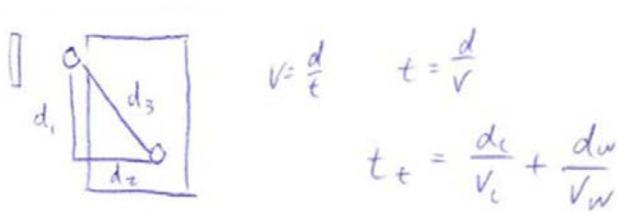


Figure 12. ***Jason's Own Model-of the Situational Problem***

Jason's empirical mathematizing didn't allow him to construct a graphical *model-of* the situational problem which he would be able to further develop into independent mathematical reasoning. Therefore, after a brief attempt to work on his own model shown in Figure 12, Jason decides to accept the journal model. Following journal guidance in construction *model-of*, he performed independent vertical mathematizing. Jason was successful in guided mathematizing, actively using the information provided in the journal.

6.3 *Nick*

Nick spent 3.5 minutes on the exploration trial and had enough time to sense the speed difference. Nevertheless, Nick's first trip used the same strategy as Kenneth's first trip - the strategy of a straight line between the platforms (the same as shown in Figure 3 (a) above). Both Kenneth and Nick, prior to their first trips, knew that the speed on land was faster than the speed in water. But their tacit intuitive model that the shortest distance should give the shortest time prevailed over knowing that the speeds were different in different media. After completion the first trip, Nick deliberately tested a few different land distances. A remarkable change in Nick's empirical mathematizing happened after the trip shown in Figure 13: he reflected in line (12), "I noticed the angle in which I enter the land from water is key in reducing the time". The 'angle' strategy trip gave Nick the best time.

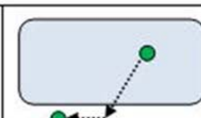
16:00	**11**	Completes the return trip:	
16:24	**12**	Transfers data about the <line 11> trip into Attempt 3 box, < T=19.18 s>; writes reflection about the trip: I noticed the angle in which I enter the land from the water is key in reducing the time.	

Figure 13. ***Fragment from Nick's Activity Transcript: the Angle Strategy***

Figure 14 shows that Nick's graphical *model-of* the situational problem, developed from his empirical mathematizing, reflects exactly his 'angle' strategy, which gave him the best time. This Nick's diagram model shows a global transition from a *model-of* his informal activity in the VE into a *model-for* formal mathematical reasoning.

Figure 14. ***Nick's graphical model-of the situational problem***

Nick was persistent in trying to develop his own model, different from the model described in the journal (see Figure 15).

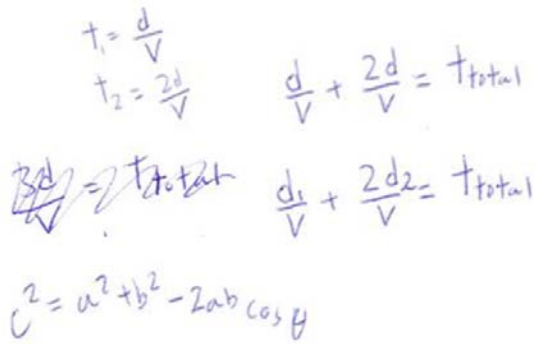


Figure 15. ***Fragments of Nick's Journal Activities***

Nick spent more than 10 minutes working on his original model independently, and another 6 minutes working on the journal model development which included guided horizontal and vertical mathematizing.

6.4 *Kate*

Kate's first trip was the best time trip. According to her comments in line (2) (Figure 16), she chose this strategy, taking into account the information that "speed is twice as fast on land". Like Jason, Kate did not take the opportunity to explore the environment before the task was assigned. Altogether, Kate used her first trip strategy of "traveling more distance on land" for 10 trips.

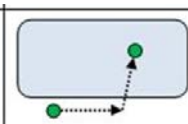
04:10	1	Completes the forward trip:	
05:16	2	Transfers data about <line 1> trip into the journal table of Attempt 1, < T=16.62 s>; writes reflections about the trip: · Since speed is twice as fast on land, traveling more dist. on land should give min. time	

Figure 16. Fragment of Kate's Activity Transcript: First Trip

Kate produced two sequential graphical models of her empirical mathematizing, as shown in Figure 17.

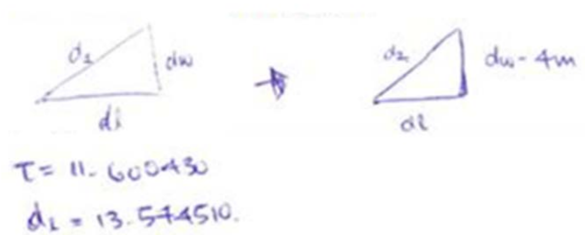


Figure 17. Kate's Graphical Model-of Empirical Mathematizing

Kate believed that her empirical mathematizing data (values of time and distance) were so reliable, that she used them for her further mathematical activity. Kate's graphical *model-of* her empirical mathematizing didn't allow her to develop a correct *model-for* correct mathematical activity, as happened with Jason. Kate failed to create her own model and, like Jason, continued mathematizing following the journal's guidance.

6.5 *Ann*

Ann spent about 2 minutes on the environment exploration and knew about speed difference. Like Kenneth and Nick, for her first trip Ann chose the strategy of shortest distance between the platforms. Ann was the third participant whose tacit intuitive model that the shortest distance between the platforms should provide the shortest time prevailed over her prior knowledge. Altogether Ann utilized a number of different strategies. The strategy of a straight line between the platforms gave Ann the best time. She rarely played computer games and it was easier for her to control the avatar along the straight line.

Ann's mathematical activity started with opening the journal pages, which provided a graphical *model-of* the situation. After reading the journal information, Ann drew the diagram shown in Figure 18.

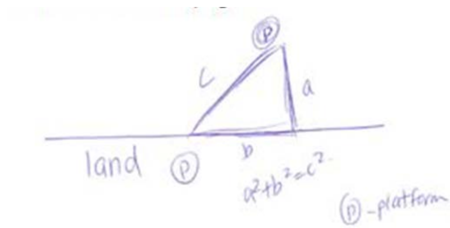


Figure 18. ***Ann's Graphical Model-of the Situational Problem***

The diagram in Figure 18 represents horizontal mathematizing; it is Ann's mathematical *model-of* the situational problem which did not allow her to develop a *model-for* correct mathematical activity. Ann preferred to accept the journal model and asked to explain it to her in detail. In the post experiment questionnaire, Ann noted that the most difficult part of the experiment was to "understanding the math".

7. Concluding Remarks

We presented separate analyses of five participants' sequential activities. Analysis of the first participant activity demonstrated the reasonability of introduction of the new term of *empirical mathematizing* as systematic planning and execution of an object-sensory activity for collecting and organizing empirical data within the problem situation for further transforming a problem field into a mathematical problem.

Examining influence of intuition on first trip choice showed that although Kenneth, Nick, and Ann knew about the speed difference from the journal and from their exploration trials, their first trip strategy was the shortest distance between the platforms. That is, their tacit intuitive model that the shortest distance should result in the shortest-time path prevailed over their knowledge about speed differences. Nevertheless, all five participants demonstrated that empirical knowledge obtained from their empirical mathematizing prevailed over their intuitive cognition/intuitive models and fully determined the *models-of* the situational problem. Particularly, the empirically obtained best-time trip strategies determined students' graphical *models-of* the situational problem. Kenneth's and Nick's best-time trips were trips that balanced land and water distances (with water distances between their minimal and maximal values) which, in turn allowed them to create *models-of* their empirical activity that were transferrable to *models-for* further mathematical development. All the other participants' best-time trips either maximised or minimised water distances, which determined their right triangle graphical *models-of* their empirical activity and were not subject to mathematical development because of their stability and absence of variables to explore. All five cases demonstrated the robustness of empirical knowledge obtained from empirical mathematizing, which in all five cases determined the horizontal mathematizing of constructing graphical *models-of* the situational problem. The assumption of this study is the following: If students are provided with opportunities for empirical mathematizing, their new empirical knowledge prevails over intuitions; their horizontal mathematizing is fully grounded in empirical mathematizing.

An important finding of this study is that empirical knowledge obtained from empirical mathematizing prevails over intuitive cognition. This is important because while 'expert' intuition can help in professional activity, lay conjectural intuition can play a confusing and contradictory role. Expert conjectural intuitions are formed on the basis of professional experience (Fischbein, 1987; Tall, 1991). Students do not have enough professional experience, so their intuitions may be lay conjectural intuitions (Fischbein, 1987). The global goal of this study is develop students' abilities to apply calculus to real-life problems (simulated in VE), which includes developing 'right' expert intuitions through making formal mathematics a natural extension of students' immediate empirical activity.

The fact that all five students, Kenneth, Jason, Nick, Kate, and Ann, developed their *models-of* the particular situational problem on the basis of their empirical activity in the VE suggests that the Second Life VE indeed provides a simulation that is close to reality; close enough to meet the purposes of this study.

The main limitation of the study is the Second Life VE programming. The educational software should be more reliable for implementing into educational practice. The computer errors, such as those which happened during Jason's VE activity, should not occur. Navigation control should not be as difficult as it was for Ann.

References

Boulos, K.M.N, Lee, H., & Wheeler, S. (2007). Second Life: An overview of the potential of 3-D Virtual Worlds in medical and health education. *Health Information and Libraries.*

Campbell, S.R. (2010). Mathematical Modeling and Virtual Environments. In R. Lesh et al. (Eds.), *Modeling Students' Mathematical Modeling Competencies*, (pp. 583-593). Springer Science+Business Media.

Davis (1996). *Teaching Mathematics: Toward a Sound Alternative*. New York: Garland Publishing.

Fischbein, E. (1999). Intuitions and schemata in mathematical reasoning. *Educational Studies in Mathematics*, *38*, 11–50.

Fischbein, E. (1989). Tacit models and mathematical reasoning. *For the Learning of Mathematics*, *9*, 9-14.

Freudenthal, H. (1991). *Revisiting Mathematics Education*: *China Lectures*. Dordrecht: Kluwer Academic Publishers.

Freudenthal, H. (1968). Why to teach mathematics so as to be useful? *Educational Studies in Mathematics*, *1*, 3–8.

Gerofsky, S. (1996). A linguistic and narrative view of word problems in mathematics education. *For the Learning of Mathematics, 16*(*2*), 36-45.

Gerofsky, S. (2006). Communication: Simulation, Reality, and Mathematical Word Problems. *For the Learning of Mathematics, 26*(*2*), 30-32.

Herscovics, N. (1996). The construction of conceptual schemes in mathematics. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 351–379). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Ilyenkov, E.V. (2009). *The Ideal in Human Activity*. New York: Lawrence & Wishart.

Lesh, R., & Zawojewski, J. S. (2007). Problem solving and modeling. In F. K. Lester (Ed.). *Second Handbook of research on mathematics teaching and learning*, (pp. 763-805). Greenwich, CT: Information Age Publishing.

Massara, F., Ancarani, F., Costabile, M., Moirano, M., & Ricotta, F. (2009). Are you telling me the truth? Authenticity of responses in virtual worlds research. *8th International Congress Marketing Trends, Paris, France*, 1-17.

Messinger, P. R., Stroulia, E., Lyons, K., Bone, M., Niu, R., Smirnov, K., & Perelgut, S. (2009). Virtual worlds - past, present, and future: New directions in social computing. *Decision Support Systems, 47*(3), 204-228.

Mueller, S. C., Jackson, C. P. T., & Skelton, R. W. (2008). Sex differences in a virtual water maze: An eye tracking and pupillometry study. *Behavioral Brain Research, 193*, 209-215.

Palm, T. (2009). Theory of authentic task situations. In L. Verschaffel, B. Greer, W. Van Dooren, & S. Mukhopadhyay (Eds.), *Modelling Verbal Descriptions of Situations,* (pp. 3-17). Rotterdam: Sense Publishers.

Pennings, T. J. (2003). Do dogs know calculus? *The College Mathematics Journal*, *34*(*3*), 178-182

Rasmussen, C., Zandieh, M., King, K., & Terro, A. (2005). Advancing mathematical activity: A practice –oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning*. *7*(*1*), 51-73.

Tall, D. O. (1991). Intuition and rigour: the role of visualization in the calculus. In S. Cunningham & W. S. Zimmermann (Eds.), *Visualization in Teaching and Learning Mathematics*, (pp.105-119). Washington DC: MAA.

Tall, D. O., Smith, D., Piez, C. (2008). Technology and calculus. In M. K. Heid & G. W. Blume (Eds.), *Research on Technology and the Teaching and Learning of Mathematics: Research Syntheses,* (vol. 1; pp. 207-258). Charlotte: Information Age Publishing.

Treffers, A. (1987). Three Dimensions: A Model of Goal and Theory Description in Mathematics Instruction. The Wiskobas Project. Dordrecht: Kluwer Academic Publishers.

Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, *54*, 9-35.