Learning to Support Young Mathematicians at Work

An Early Algebra Resource for Professional Development

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Acknowledgments

Materials Development

Mathematics in the City

City College of New York

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- Despina Stylianou, Co-Principal Investigator
- Kara Imm, Research Assistant, Professional Development Materials
- Herbert Seignoret, Staff Assistant

Freudenthal Institute

Utrecht University, The Netherlands

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1The authors gratefully acknowledge the many teachers who agreed to be part of the field-testing and research on the use and effect of the materials.
Research and Evaluation Partners

- Horizon, Inc., Chapel Hill, North Carolina
  - Daniel Heck, Director
  - Courtney Nelson, Assistant Director
  - Murray Wickwire, Researcher
  - Gwen Moffett, Researcher
  - Shayla Thomas, Researcher

- Eve Torrence, Randolph-Macon College, Virginia, Outside Evaluator

- Fall River, Massachusetts Public Schools
  - Fran Roy, Assistant Superintendent, Providing a Research Site

Video Production

- Catherine Kellison, Roseville Video
- Jeffrey McLaughlin, Editor

Video Crew

Human behavior, whether mental activity or overt movement, is the product of many interacting parts that work together to produce a coherent pattern under particular task, social, and environmental constraints. I see development as this process of assembling patterns of behavior to meet demands of the task in the biological possibilities of the learner at that time. Sometimes the behaviors are stable for a time—they are easily elicited and frequently performed. Then other behaviors emerge, and the old ones become less available or less preferred. The critical issue for development is to understand how activity on short time scales cascades into the behavioral changes in which we are interested. It may be, for example, that we cannot draw a line and call one process learning, another development, and still another a therapeutic intervention; it is all change over time.


The decisions and actions teachers make in the heart of teaching are often split-second reactions made in the moment of interacting with their students, and thus inherently are a result of their subconscious (Dolk 1997). They are a product of beliefs about teaching and learning, a result of perceptions, affected by emotions and the climate and culture in schools, and constrained by the particular tasks teachers are called on to do and the assessment policies that are mandated in our institutions. They are even somewhat determined by memories of teachers’ own past schooling. To the point, cognition as it relates to teaching is “embodied” (Varela, Thompson, and Rosch 1991; Thelen and Smith 1994). This fact makes our work as teacher educators and math coaches difficult. “What kind of effect can we have with just a workshop, or a course or two of study?” we ask.

If cognition is embodied, the context of the elementary classroom is critical for teacher education. So, how do we effectively educate teachers in a workshop setting, or from a university classroom? The traditional approach has been to teach methods courses and send teacher candidates to schools for fieldwork assignments and student teaching. Another approach, more popular in the professional development circle, has been to utilize a “lesson study” model. Here a group of teachers plan a lesson, one carries it out with the group observing, and then a “debrief” ensues.
Both of these approaches, however, leave much to be desired. Perception itself is embodied—we see initially what we expect to see (Thelen and Smith 1994). To effect change, teacher educators need to: (1) bring up participants’ initial observations; (2) analyze and discuss discrepant, even contradictory perceptions; and (3) encourage relooking again, and again, to seek evidence for interpretations. To do these things, we need ways to rewind time, analyze moments in depth, examine the work of a class and individual work of children in that class over time, view instruction and learning over a sequence of activities (not just one lesson), and examine and discuss teachers’ conferrals and their questioning in relation to later work of the children.

Beginning in 2000 with funding from the National Science Foundation (NSF), we began to develop a digital library of professional development materials. These materials currently comprise a library of seventeen CD-ROMs and fifteen facilitator guides on number and operation (see Young Mathematicians at Work Resource Packages). They are published by Heinemann and widely used by teacher educators, math curriculum coordinators, and math coaches. The materials you hold in your hand extend this library to include two DVDS on early algebra.

Digital production projects have traditionally used the technology to accompany texts, for illustrating examples such as lessons, coaching models, and/or interviews, or in other cases for modified lesson study (for example, see www.Lessonlab.com). Our digital lab environments have different purposes. Each is a professional development course featuring several classroom sessions in which the class explores one particular mathematical topic in a sequence of activities now published as units in the series Contexts for Learning Mathematics (Heinemann). Users can study children over time in these classrooms; they can examine the teacher’s didactical employment of context and inquire about and analyze pedagogy. They can clip and paste moments from footage and build learning trajectories (or “landscapes of learning” as we prefer to call them, which show children constructing big ideas, developing strategies, and/or using mathematical models as tools). This approach provides an active, more meaningful professional development experience, which empowers teachers to integrate theory and practice. Teachers also deepen their own mathematics content knowledge as they engage in investigations outlined in the facilitator guides; solve mathematical problems in several ways and anticipate student strategies, which they subsequently examine; design investigations and minilessons for the next day; subsequently examine how the teacher in the environment continues; and analyze children’s work to assess the effectiveness of the instruction. Users can even add clipped footage as hypertext evidence to support arguments and provide examples in term papers and literature reviews, or as sample evidence of the National Council of Teachers of Mathematics (NCTM) standards, Focal Points, or the Common Core State Standards. The materials enable users to work at home on assignments as well as in college classrooms since focus questions and notepads are built right into the environments. They can also be used in Internet study groups, in summer institutes, in college classrooms, or in workshop settings. (For information on offerings using the materials by the authors see www.newperspectivesonlearning.com and/or www.mitcccny.org.)
The accompanying facilitator guide is organized for you in a variety of ways. Material is first provided about loading the program onto your desktop and getting started. If you are planning on participants in your workshops also working at home, you (or they) may purchase individual copies of the DVDs, separately.

Next, you will find a description of how the materials are used in two of our Mathematics in the City (MITC) NSF-funded five-day summer institutes (one based on *Trades, Jumps, and Stops* [Fosnot and Lent 2007] and the other on *The California Frog-Jumping Contest* [Jacob and Fosnot 2007]). Each day is described in detail with suggested activities and their purposes. Materials needed and related readings are also listed. Throughout there are sample dialogue boxes to help you envision what your participants might say. A side column contains notes from one of the authors. In a sense, as you read and picture the workshop and participant interaction, the authors co-teach with you, sharing their experiences and insights as teacher educators.

Sometimes we don’t have the luxury of doing a one-week summer institute. In the next section of the guide, you will find detailed descriptions for several sample shorter workshops that can be done in a day, or in two to three hours. These are workshops on specific topics related to algebra; for example, structuring, variation, equivalence, proof, or the role of representation.

**The Landscape of Teacher Development**

In previous volumes and curricular units, we suggested that learning can be conceptualized as dynamic movement along an undulating terrain—a landscape of learning (Fosnot and Jacob 2010; Fosnot and Dolk 2001, 2002; Fosnot et al. 2007). In contrast to a single, linear pathway that all learners march along, we see learning as horizontal and vertical movement resulting in important developmental landmarks as learners try out new behaviors and construct meaning. These landmarks include big mathematical ideas as learners attempt to structure part–whole relations (such as the distributive property, variation and equivalence, or the relationship between addition and subtraction). They also include progressive schematization (for example, the move from skip counting and repeated addition strategies to regrouping the groups into partial products; or the movement from trial and error, to arithmetic strategies, to algebraic strategies) and emergent modeling (from models as representations of contexts, to the use of generalizable models such as number lines and arrays as tools for thinking).

We characterized learning graphically this way so that teachers could do two things: first, situate their students developmentally, and then recognize and support important moments or shifts in children’s mathematical development. It is helpful to know not just where children are, but what is just coming into view on their mathematical horizon.

But children are not the only ones engaged in a developmental process. Their teachers are also traveling along a different, but related, landscape. While their students are developing new ideas about mathematics, teachers are also developing new ideas about the content (for example, what algebra
is, what it means to do algebra, and how this capacity develops) and about pedagogy (for example, what teaching techniques might be helpful, how best to question and confer, and when to provide appropriate reflection and practice). They are deepening their ability to “kidwatch” and connect what they are seeing to mathematics development, and they are developing new ideas about didactics (for example, what effects the choice of numbers and context have on development, and how context can be purposefully crafted to ensure powerful learning).

We began by researching and documenting general teacher development in relation to number and operation along three domains: mathematics, pedagogy, and didactics. We asked ourselves, “As teachers begin to deepen their practice, what significant shifts are we likely to see? What ideas about the teaching of mathematics will feel new and perhaps strange to them? How will their pedagogy change? How will their use of contexts and models shift?”

In general, we found that as teachers developed a deeper understanding of number and operation and became more aware of the associated landscape of learning, they were better able to notice the emergence of important ideas and strategies in children’s attempts and maximize important teaching moments. The teachers shifted from a mechanical use of context merely as a locus for applying taught procedures toward the use of (realistic: realizable, imaginable) contexts and “truly problematic” situations both as starting points for mathematical constructions and as a teaching tool to facilitate mathematical development. Simultaneously we noticed shifts in pedagogy from teaching by explanation, practice, and reinforcement to teaching to facilitate students’ dialogue, mathematical inquiries, and cognitive constructions (Fosnot 2000, Dolk, et al. 2000).

Our specific findings in relation to the teaching and learning of algebra are more tentative, since our research on the use of the algebra materials is still underway. But we present our working hypotheses and questions here, because we believe they will help you to support the development of participants in workshops and courses as you use the materials. We are examining what conceptions (and misconceptions) about algebra participants have already developed and how these change as they work with the materials. In particular, we are hypothesizing (and noticing) that growth is likely to occur in a number of areas:

- views about of algebra and proof
- questioning and conferring
- kidwatching
- the use of context
- the use of representation

In the sections that follow, we expand on each of these ideas in greater detail and provide a description of these interconnected hypothetical landscapes.

Like the landscapes we have developed for young learners, these landscapes are not meant to serve as checklists or rubrics of what teachers should know and be able to do. Nor are they meant to categorize teachers into those
who “get it” and those who don’t. Rather, the landscapes serve as possible working road maps of development with various pathways—a model of the important developmental shifts we hope to support as teacher educators.

**Views About Algebra and Proof**

We have found that most participants in our workshops initially describe algebra as the strand of mathematics where letters are used and they are hard-pressed to go beyond this description. So cemented is this belief, that when asked to think about why Aisha (the teacher on the *Trades, Jumps, and Stops* DVD) does not use letters when recording the discussion on the number of quarters in the ten-dollar wrapper, they often state, “I would write $4q = 1d$ to introduce more algebra.” When asked to write an equation that represents the relation between the *number* of quarters and the *number* of dollars, they in fact write the same equation, $4q = 1d$, unaware of the reversal error, and when they plug in numbers they can’t understand why their equation doesn’t work. (For example, if they say the number of quarters is 8, they would get $4 \times 8 = 1 \times 32$, or 32 dollars, which is obviously wrong.) Often participants need much disequilibrium and cognitive reorganization before they are willing to accept that the equation should be $q = 4d$.

As they work with *The California Frog-Jumping Contest* DVD, they often assume erroneously that the length of a frog’s jump and the length of a step are both variables. The length of a frog’s jump is a variable and can be denoted with a $j$ to represent that it varies from frog to frog, but the length of a step is a constant—a specific unit of measure used by the referee. The length of the step does not vary.

One of the important big ideas about algebra that surfaces as participants work with the materials is that equations show relations, in contrast to procedures with the answer coming after the equal sign, and they begin to recognize the difference between an arithmetic strategy where the value of $c$ in the equation $3 + 7 + 14 = 7 + 3 + c$ is found by adding up all the terms on both sides of the equation, left to right, and a more algebraic one employing commutativity ($3 + 7 = 7 + 3$) and treating the expressions as equivalent objects that can be canceled, thereby producing immediately the equation, $14 = c$.

Their understanding of variation also deepens. They may at first use letters as specific, familiar nouns (for example, thinking of $q$ as a quarter with a specific value, $q = .25$). This use is why they often have trouble writing equations and make reversal errors. Also, when a letter appears in an equation, participants seem to have a singular mission: find its value. As they work with the materials, conceptions of the multiple meanings of a variable will develop (for example, to express relationships, to show quantities that vary, as an object that can be operated on even when the value is unknown, as a function with two variables $48 = 65 + 8E$, and in a system of related equations—for example, when exploring frog jumping data, they are surprised to discover that $4j + 8 = 52$ can be used to determine the value of $2j + 4$, without having to solve for $j$ first; the value is half of the first expression).

Descriptions of proof can also be insufficient. In many classrooms, proof is understood and used as a general request to “show your thinking.” The activity of justifying one’s thinking is trivialized to “Tell me in words, pictures,
and symbols what you did." This raises the question for any mathematical community: What is the standard by which we accept reasoning as proof? This standard, of course, varies depending on the mathematics and the development of the students, but the process of proof-making is developmental and thus it makes no sense to wait until college to teach a course on it, as if one course would be sufficient!

Professional mathematicians don’t just “solve problems.” They craft proofs of their solutions for others to read. To meet the Common Core Standards of Mathematical Practice—treating children as “developing mathematicians”—demands that we provide opportunities for them to publish their arguments to convince their peers. Certainly we don’t expect children to construct the kinds of formal written proofs that professional mathematicians produce, but that doesn’t mean that students’ reasoning isn’t a vital part of the mathematics we expect from them. We do want to support young mathematicians to:

- articulate their thinking verbally and/or in writing in such a way that others in the community can make sense of it (not just the teacher)
- help them formalize and systematize their reasoning so that it can be generalized
- defend and revise their reasoning when others challenge it
- begin to understand what a piece of written mathematics looks like

Initially participants may argue, “I’ve proven something because I’ve tried a procedure many, many times.” When the question is, “Did it work? How do you know?” the response may be, “Yes. I know I’ve found all the combinations because I’ve tried a bunch of them and I can’t find any more.” But efforts may be random and not strategically organized.

Next, we may see the use of a systematic approach. Here’s an example from a teacher participant we’ll call Ricky. “I know there are only four ways to make 10 cents with dimes, nickels and pennies. First I used only dimes and that generated one way. Then I used only nickels and that gave me a second way—2 nickels. Next I used only pennies and that was a third way—10 pennies. Last I started to trade some nickels for some pennies and I got the last way—1 nickel and 5 pennies. That is the only swap I can make, so I know I’ve found them all.” The introduction of systemization (like Ricky’s approach) elevates the reasoning as it represents a cognitive reorganization—a shift away from just trying combinations (a guess and check approach) to using a list of all cases.

Later, proof may include the explanation for how a particular mathematical strategy or solution can be generalized. For example, we might ask, “Does Ricky’s approach to finding the possible combinations of dimes, nickels, and pennies work for other amounts? What about different combinations of coins? How can we prove we have all of the combinations?” Questions like these move from the specific case to the general case—an important shift in the development of thinking about proof.

Last, one key element of proof is the development of deductive reasoning. One form of this reasoning can be seen in “if–then” statements, though there are other forms that deduction can take. Often we see participants initially use circular logic, beginning with the assumption they are trying to prove. Starting with givens, defining one’s terms, and building up to what
one is trying to prove feels backward to them. As teachers’ notions of proof expand, we may see the following:

- progression from seeing proof as a record of “what I did” to a logical and complete mathematical argument
- the recognition that writing a convincing proof will likely require many drafts and revisions
- using proof by induction or proof by contradiction
- a movement away from formal two-column proofs as the only model for proofs (Note: Though most teachers associate proof with these relics of high school geometry class, many professional mathematicians do not even consider them actual proofs since they often lack full mathematical ideas [either in sentence or in symbolic form]. Instead, two-column proofs might be considered an outline—an incomplete product on its way to becoming a finished detailed argument. Still, they are very helpful as drafts.)
- a general move toward arguments that are elegant, simple, thorough, concise, creative, clear, efficient, and possibly even clever

As they change their own beliefs about proof, participants begin to make changes in their practices:

- They invite learners to write up their arguments and provide gallery walks where students review the texts and make helpful editorial comments to the authors.
- They provide time for revising and publication.
- They encourage learners to generalize and write about important ideas instead of just writing about what they did.
- They provide time for discussion of ideas, for example, in math congresses, and facilitate the discussions in ways that foster deep reflection and inquiry.

Figures 1a and 1b provide graphics of the landmark changes you might see as participants in your workshops progress on this journey.

### Questioning and Conferring

There is no shortage of volumes for teachers about the importance of questioning in one’s teaching practice. Perhaps as a result few teachers would disagree with the idea that good teaching includes some aspect of skillful questioning. Beyond this, however, their sense of the role of questions may be limited. Many participants originally may possess an unexamined belief in the importance of questioning, but they do not know what to ask, when to ask a question, or for what reason. One of the most commonly requested workshops, in fact, is on the role of questioning. This request, though, often comes from the fact that participants just want to exchange one teacher behavior (explanation) with another (questioning). Since explanation was assumed to be capable of transmitting intended outcomes, these teachers assume that questioning should be used similarly. Given ongoing national reform in mathematics, many teachers have come
Equations show relations

Numeric expressions are treated as objects

Flexible use of multiplicative structuring

Substituting and exchanging equal expressions

Symbolizing relationships with variables (without the reversal error)

Equivalence is a big idea in algebra

Variation: variables can be used to describe relationships

Algebraic expressions can be treated as objects

Uses additive, but not multiplicative, structuring when looking for patterns

Equations show relations

Uses equivalence to separate off equal expressions

Variables are used to represent unknowns

Algebra is the strand of mathematics where letters are used

Algebra is about the development of structuring

Algebra is about generalizing

Make the reversal error

Undoing and arithmetic strategies are used to solve for unknowns (emphasis is on procedures)

Figure 1a Views about algebra
Proof can take the form of deductive reasoning. Proof entails moving beyond a specific case toward an explanation of why and how it can be generalized. Describes an example with recognition of its generality.

There are many types of proofs (induction, deduction, proof by all cases, etc.).

A proof is rigorously constructed. No claim is treated as “common sense,” “commonly understood” or “obvious.”

Works for clarity in definitions and processes.

A proof can be written in complete sentences and is not necessarily a formal, two-column one like those usually produced in a geometry class.

I can disprove something by finding just one counter example. But to prove something a single example is not enough, nor is a list of examples.

Retells a solution process.

Proof means convincing yourself you are right—being sure.

Proof moves beyond the particular problem and involves constructing a mathematical theory.

Recording Generalizable Observations (Universal generalization).

Modus ponens

Proof by all cases

A proof may be based on a systematic approach that demonstrates that you have exhausted all possibilities.

Restructuring involves a new chain of reasoning.

Analyzes and resequences a process.

The standard for a proof is convincing other people that your thinking makes sense.

Figure 1b  The development of proof
to understand that “telling” students the mathematics they want them to understand is not ideal. In light of this trend, teachers may avoid the explicit telling of mathematics and look for questions instead to lead students toward the strategies or answers they want to hear. In essence, this early use of questioning serves as a proxy for a more direct effort to transmit some mathematics to the students. For example, a teacher might ask, “Were you subtracting 3 from both sides in order to keep the equation balanced?” While teachers who frequently use questions like these may have a clear purpose for their use, it is usually to push students toward their own mathematics. As an alternative, we want to move toward the kinds of questions that allow us, first and foremost, to understand students’ thinking, to ensure that this understanding is shared by the other young mathematicians in the classroom, to foster reflection and disequilibrium (when needed), and to generate further inquiry and generalization.

Sometimes the first shift teachers make is asking more open questions, but it’s almost like a recipe. They ask the same thing no matter what the child is doing. They start with, “What strategy did you use?” Then ask, “Are you sure you are right? Can you prove it?” They seem to have constructed the idea that talking about mathematics has value and therefore that should be the goal of every conference—get students to talk about what they did. Realizing that talking is important is an advance on the landscape, but still on the horizon for these participants is the idea that conferring only starts with an understanding of a student’s thinking so that the teacher can determine what’s on the horizon, where to go next, whether to foster development of a big idea just coming into view, or whether to focus on a refinement of the approach. A good conference is more about the development of the young mathematician than it is about the arrival at an answer to the problem, or correcting the mathematics on the page.

It is tempting to believe that good teachers have a magical list of questions that they draw from, inserting them into conversation whenever the teachers deem them useful. Those who have done professional development work with teachers are sometimes asked to provide a “list” of questions so that teachers can emulate their style of questioning. Or, teachers often mention that if they knew what questions to ask (and when), their teaching overall would improve. This raises two related issues: whether questioning is really about style, and whether inserting questions from a predetermined list could ever improve one’s teaching. We believe that neither adopting a new style nor relying on a list of questions will ultimately result in a shift in practice. Instead, the shift occurs when teachers believe that the development of useful questions comes from understanding the specific mathematical development that they want to encourage, probing to understand a learner’s thinking, helping inconsistencies or “dead ends” become apparent, encouraging inquiries, and pushing for generalization and proof. So understanding the landscape of development for young learners and a deepening of one’s own content knowledge enable teachers to become better questioners.

Participants often ask of us, “What questions do I ask to get children to be able to solve the problem?” They may believe the purpose of questions, in essence, is to pull out or extract some predetermined idea or answer—usually theirs. They may ask leading questions like, “Do you think adding all the coins up to see what the total is would be helpful?” They often view Aisha’s early questioning on the DVD during the Masloppy investigation as positive
simply because she is asking questions and then are surprised when they read her journal reflections on her questioning. They also have a difficult time analyzing the importance of the questions she asks during the game of Twenty Questions. Moments like these are critical for participant learning, and several questioning and conferring activities are built into the digital environments and are discussed later in sections 2 and 3 of this guide.

Eventually, teachers will come to use questions in savvy and sophisticated ways. For example, questions can be used to promote cognitive disequilibrium—that is, they generate some confusion or puzzlement about a mathematical idea. This particular use of questions serves to create some deeper reflection. For example: “We’ve just heard Jonah’s and Ramon’s conjectures, but they are each saying something different. Could they both be right?” Or, “Is it possible that at least one bag wouldn’t have two quarters?” Or, “I wonder if there is any way to prove these are equivalent without doing all of the arithmetic?” Becoming able to ask powerful questions is directly related to understanding what it means to do mathematics, wanting to support the process, and deeply understanding the development of the mathematics content at hand.

Figure 2 provides a graphic of the landmark changes you might see as participants in your workshops progress on this journey.

**Kidwatching**

Often when participants are asked to observe and take notes on a pair of students at work, they focus initially on affect, involvement, and other non-mathematical issues. They comment on who’s working, who’s on task, who talks more frequently, and whether the children are cooperating. They need to be pushed and focused to examine the mathematics. Even when seeing a clip for a second time, they often disagree on what they see. These moments are powerful. It is through the disagreements, the sharing of the different lenses of participants, and the revisiting of the clips that teachers come to see with more insight. As humans, we interpret what we see quickly as a way to make sense of and process the visual stimuli efficiently. Encouraging participants to seek evidence of their statements about a strategy they think children are using, before interpreting and inferring what they think they know, is critical to their growth as teachers.

The first shift that occurs is the ability to watch a pair of children at work, label the strategy, and when probed produce evidence that supports the conclusion. It is much more difficult for participants to see the underlying big ideas or the mathematics just out of reach but coming into view. For example, they may see that Diana (in the Masloppy investigation on the DVD) is substituting and exchanging equivalent amounts as she finds ways to make use of the nickels (an important algebraic strategy) but miss the fact that she is close to a systematic proof and with some support could be encouraged to see that the problem can be narrowed down to what to do with the nickels, leading to a proof by listing all the cases. A deep understanding of the mathematics and the associated landscape of learning affects teachers’ ability to be good kidwatchers.

Figure 3 provides a graphic of the landmark changes you might see as participants in your workshops progress on this journey.
Questions are used to foster reasoning and proof.

Questions and invitations are used to foster generalizations beyond specific contexts.

During conferring, teacher offers suggestions and refinements as wonderings, invitations, and possible pursuits.

Questions are used to promote disequilibrium, puzzlement, and cognitive reorganization.

Students’ questions become the focus of areas of inquiry for individual students as well as whole group conversations.

Questions are focused on creating a chronology of the student’s process. (What did you do first? Then what? Why? What next?)

A general set of questions is routinely used (How do you know that? Can you prove it?). Questioning is not connected to specific mathematics.

Questions should probe thinking and engender reflection.

Uses guiding questions to get students towards the teacher’s answer and solution strategy.

Teaching is about mentoring and initiating students into a community of practice. Development

Questions facilitate development, though which questions to use and for what purpose remains unclear.

Teacher asks questions and students answer them. Purpose is to see what students know and to keep them engaged.

Questions motivate and keep students engaged.

Questions are largely focused on mathematical procedures (e.g. What do I do with my X?).

Few questions are present. Teaching is largely done by telling.

Questions are largely used to generate the recall of mathematical facts or vocabulary.

Teaching is about transmitting information and explaining concepts.

Teaching is about facilitating development.

Questions and invitations are used to foster generalizations beyond specific contexts.

During conferring, teacher offers suggestions and refinements as wonderings, invitations, and possible pursuits.

Questions are used to foster reasoning and proof.

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Students’ questions become the focus of areas of inquiry for individual students as well as whole group conversations.

Questions are focused on creating a chronology of the student’s process. (What did you do first? Then what? Why? What next?)

A general set of questions is routinely used (How do you know that? Can you prove it?). Questioning is not connected to specific mathematics.

Questions should probe thinking and engender reflection.

Uses guiding questions to get students towards the teacher’s answer and solution strategy.

Teaching is about mentoring and initiating students into a community of practice. Development

Questions facilitate development, though which questions to use and for what purpose remains unclear.

Teacher asks questions and students answer them. Purpose is to see what students know and to keep them engaged.

Questions motivate and keep students engaged.

Questions are largely focused on mathematical procedures (e.g. What do I do with my X?).

Few questions are present. Teaching is largely done by telling.

Teaching is about transmitting information and explaining concepts.

Teaching is about facilitating development.

Questions and invitations are used to foster generalizations beyond specific contexts.

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Figure 2 The development of questioning and conferring

30 LEARNING TO SUPPORT YOUNG MATHEMATICIANS AT WORK
Kidwatching should be connected to a deep knowledge of the mathematical landscape. Inferences about what a child knows are made as a result of evidence. Kidwatching focuses on students’ strategies, but inferences are often made without evidence. Purpose of kidwatching is to see if they are working well and if they are going to be successful in arriving at the teacher’s answer. Kidwatching takes into account how representations are being used.

Kidwatching takes into account the logic in a child’s argument. Kidwatching begins to consider emergent big mathematical ideas underlying the strategy, and on the child’s horizon.

Kidwatching takes into account gestures that may have mathematical meaning. Kidwatching is focused on evidence of students’ strategies, noticing what they did or said.

Purpose of kidwatching is to see what mathematical strategy they are trying. Kidwatching is focused on affect—who’s working, who’s on task, who talks more frequently, etc.
The Use of Context

At least some participants in our workshops initially use context rarely, if at all. Instead, they may focus on tasks and problems that are context-free (“bare number” work, containing only numerals and other mathematical symbols). For example, they may present their students with a myriad of “missing addend” problems, such as $9 + x = 15$, as a way of doing algebra, and the classroom conversation tends to focus on the mathematical procedures used to solve the problems.

Even when teachers first begin to use context in their practice, they may do so in a variety of insufficient ways. At first, they may:

- Believe that problems with context should occur only at the end of a unit of study, as a way to practice procedures or apply some mathematics that has already been mastered. Thus, they design problems for students to apply their prior learning. Example: “We’ve been doing mystery number problems, so now you are ready to tackle these problems about George’s allowance. He had saved $9. After his parents gave him his weekly allowance he had $15. How much did his parents give him?”

- Consider the context as a place to “house the numbers.” That is, the purpose of the context is to contain the values and operations that they want their students to use.

One of the first important developmental shifts to notice is from the use of context as an ending point (word problems are given for application of previously taught procedures) toward the use of context as a starting point (to generate the mathematics). Participants often believe initially that students must first be shown or told about a strategy before they can be successful using it on their own. “How could they be ‘ready’ for this problem when we haven’t shown them the strategies to use?” teachers often ask. This type of thinking rests on the idea that students are not likely, or able, to generate strategies of their own. In contrast, we believe that students do, in fact, possess problem-solving strategies and are capable of generating algebraic ideas, and so, when we begin a unit of study with a context-specific investigation, we give students a chance to generate a myriad of new strategies. As participants in workshops study the clips on the DVDs of children at work and listen to the discussions in the math congresses, they are often amazed at the children’s insights, sometimes even commenting that their children could not do the kind of thinking they are witnessing. Bringing these beliefs to the surface in discussions and examining how the contexts on both DVDs are supporting children is critical for powerful learning.

Conceptually, the purpose of the context may begin to shift as well. If contexts are seen only as places to store numbers, then they can be relatively generic—interchangeable and inconsequential to the mathematics itself. Choose any context (for example, a trip to the deli, a cell phone plan, planning a birthday party), the thinking goes, because it is the numerical and symbolic values, not the contexts, that are vital. From this perspective, the contexts chosen are often just trivialized word problems. A shift occurs when the context is chosen because of, and in light of, the mathematics. In textboxes throughout the DVDs, and described in the guide, are several teaching moments where...
participants are asked to consider the contexts employed, how they elicit a host of strategies at the start and then ensure progressive development, and the models they generate (such as the double open number line).

Eventually ideas about the use of context shift further. Instead of seeing contexts only as a starting or ending point, teachers use context didactically—crafted to foster the development of mathematical ideas, and in particular to support development along the related landscape of learning (described on page 12 of *Trades, Jumps, and Stops*, and page 14 of *The California Frog-Jumping Contest*—the Context for Learning Mathematics [CFLM] units under study in this set of materials). By carefully choosing numbers, building in constraints that lead students to abandon inefficient strategies and generate new strategies or creating situations for specific models to emerge, teachers come to understand context as a vehicle to foster development. For example, the choice of numbers in the Masloppy Piggy Bank context in the *Trades, Jumps, and Stops* unit (the number of wrappers can easily be distributed and thus disregarded) supports children to consider the algebraic strategies of canceling and substituting equivalent expressions (trading dimes for nickels, etc.) rather than employing arithmetic (for example, adding everything up). The context is also flexible enough to accommodate increasingly more complex mathematics in that more than one possible solution exists and children can potentially examine all possible cases and come to develop a systematic way to approach the problem and prove their thinking to others. The Sunny, Cal, and Legs investigation in *The California Frog-Jumping Contest* unit supports students to represent expressions in relation to one another to deepen an understanding of equivalence and to determine unknown values by canceling out equivalent pieces.

Finally, participants come to appreciate the need for sequence (in contrast to single activities) in order to ensure development over time. Instead of providing students with a few isolated investigations, participants come to realize the importance of developing entire sequences, employing several contexts. They come to appreciate the sequence of investigations in the unit, the related minilessons, and how the use of context throughout supports both vertical and horizontal mathematizing over time to develop a network of related ideas, as well as the development of algebraic structuring in general (for more detail on this see Fosnot and Jacob 2010).

Figure 4 provides a graphic of the landmark changes you might see as participants in your workshops progress on this journey.

### The Use of Representation

Mathematical representations are often initially seen as “artifacts” of a person’s thinking: an external, often physical relic of some internal, individualized, and unseeable process. “How did you solve this problem?” is often perceived as an invitation to produce a written representation, even if the representation itself does not mirror exactly the thinking. In this way, representations are considered products of thinking, not the thinking process itself.

Because our thinking is largely inaccessible to others, representations provide some insight into what that process might have been. So, “Show me your representation” can mean simply “Write something down in a way that helps
The contexts used are usually trivialized word problems—stories that may or may not be closely related to the mathematics at hand and/or crafted tightly to ensure that the teacher’s desired strategy will arise. Contexts are intentionally crafted to foster the development of strategies and the landscape is used as a framework for development. Contexts can be used before procedures are taught to generate a discussion of strategies. Learners are asked to solve word problems to generate mathematics, but the contexts are not crafted to ensure development. They are used only as a locus for generating learners’ current knowledge. Contexts are used to practice a skill or strategy already taught. Word problems are used for application of previously taught procedures. Mathematics is about understanding how and why procedures work. Context provides a space to apply these procedures. No context used at all. Focus is on symbols and procedures only.

Development involves both vertical and horizontal mathematizing. Contexts are crafted in a sequence to foster progressive development involving both horizontal and vertical mathematizing. Context can be crafted to promote the progressive development on the landscape.

Figure 4 Use and purpose of context
me to understand your thinking.” But mathematical representation includes both the product of one’s thinking (the physical artifact) and the process of representing. Modeling is critical to the process of mathematizing. That is, one represents whether or not there is a physical document to demonstrate this process. In fact, there are perhaps two related processes at work: how a person represented an idea cognitively (how models were used as tools), and then how they reproduced that idea on paper, usually for others. When the conception of representation moves from product only to both product and process, this suggests a major shift in a teacher’s thinking.

Teachers use mathematical representations in a variety of ways. Many participants in our workshops initially believe that students’ struggles to solve a particular problem or use a particular strategy might be resolved if only they could “see” the mathematics. This “seeing” of the mathematics rests on the belief that mathematics will be revealed to students when particular representations are used. The teachers imagine that simply by showing students a balance beam or scale, students will come to understand the idea of equivalence, or that they will “see” place value relationships by using base ten blocks. But, as they eventually come to learn, the mathematics is not in the model to be seen—it develops in the learner’s mind and is only then brought to bear on the models and representations.

Teachers often say, “Draw a picture or show in words or symbols what you did.” This trivializes the notion of mathematical representation and can be puzzling to students since this additional representation may not be useful to them. They’ve already solved the problem in a way that makes sense to them, and they receive no new insights by now drawing a picture. They are doing it only to please the teacher.

In previous work, we have discussed the importance of models for developing algebra (Fosnot and Jacob 2010). Models are one especially powerful form of representation. A teacher’s ability to use appropriate models to represent a student’s computation strategy is an important first step. This allows students to have images of their strategies to discuss and extend. But this assumes that students understand the model itself. Typically the use of models occurs in three stages depicting a movement from “models of” to “models for” (Gravemeijer 1999):

1. Models are first introduced as a model of a realistic situation (for example, the use of a double open number line to represent equivalent distances on a jumping track).
2. Models are used as a representation of a student’s computation strategy (for example, a teacher might show a student’s addition strategy as jumps on a number line).
3. Models become tools for thinking—students themselves use models as tools for thinking about new problems unrelated to the contexts used to develop the models initially.

In classrooms where models are used frequently, important conversations about the relationship between representations often emerge. What mathematics is illuminated or constrained in the use of one model over another? How does the choice of representation support students’ development mathematically?
As participants work with the materials, ample opportunities will occur to examine the powerful role of representation in mathematics. For example, the double open number line can serve as a powerful model for developing equivalence. Also, a variety of representations can represent the relationship between the number of quarters and the number of dollars, for example, ratio tables, graphs, and equations. As you use the materials, look for ways to encourage participants to examine alternative representations and the power of each.

Figure 5 provides a graphic of the landmark changes you might see as participants in your workshops progress on this journey.

Teacher development is very complex and embedded—directly linked to the context of the classroom. The pathways through each of the domains described are multiple with many connections, and growth in one domain often has ramifications for growth in others. Overlaying the pathways of the five domains produces a representation of the complexity of teacher development as the domains are in truth not separate. As you work with participants with the materials, look for moments where change in one domain produces a change in another. These are important teaching moments. Focus discussion and reflection on them.

One-shot professional development workshops, even when they are a week in length, will not likely be powerful enough to provide for sustained movement along this landscape. Precisely because teacher learning is embedded and occurs in the moment of decision-making in the act of teaching, it is important for participants to use the related Context for Learning Mathematics units in their classrooms and for in-class support to be provided. Whenever possible, this should be in the form of co-teaching. In our estimation, cognitive- or content-based coaching in the form of pre- and postdiscussions on the intent of the activities, while possibly somewhat beneficial, will never be powerful enough to effect sustained change, as learning is not likely to transfer beyond the use of the materials. In contrast to pre- and post-discussions, co-teaching provides help in the split second of the decision-making as the teacher formulates questions and facilitates discussions. It is through this coacting and cothinking and subsequent reflection and collaborative discourse that powerful new learning emerges.
Representations are written records of a person’s thinking (e.g. products).

Representations are used to show students the math, usually that they are struggling to “see.”

Multiple representations are presented by the teacher so that students of a variety of “learning styles” will “see” the mathematics.

Asks students to represent solutions. “Can you represent that?” usually means “Can you draw a picture of that, or show me in words, or symbols what you did?”

Mathematics can be either illuminated or constrained depending on the choice of representation to be used.

Multiple representations are presented by the teacher. The relationship between the representations and/or strengths and weaknesses of each are explored.

Multiple representations are developed in sequences as tools for thinking.

Representations are used to support the development of mathematical ideas (e.g. use of the double number line to support equivalence).

Representations are vaguely understood to be an important feature of the teaching and learning of mathematics.

Representations are used to model students’ strategies, especially for computation.

Representations can be used as tools for thinking.

Representations go through stages of development.

Figure 5 The development of the use of representation
References


