

The FOOTPRINT PROBLEM: *A Pathway to Modeling*

By exploring an open-ended investigation involving proportional reasoning, students were able to walk through both problem solving and modeling.

Kara L. Imm and Meredith D. Lorber

It is significant that the fourth Standard for Mathematical Practice: Model with mathematics (CCSSI 2010, p. 7) is part of the Common Core State Standards for Mathematics (CCSSM). It is noteworthy for two major reasons. First, never before has mathematical modeling played such an explicit role in K–grade 12 mathematics standards. By elevating modeling to the place of mathematical practice, CCSSM invites teachers to consider what modeling entails. Second, the descriptions of each of the mathematical practices are decidedly brief, leaving open the possibility for

wide interpretation. Because the term *modeling* is used in a variety of ways in education outside of a mathematical context (e.g., the demonstration stage of a lesson, the construction of some concrete or physical solid, the hopeful emulation of a specific behavior), the term is ripe for misinterpretation. As a Mathematical Practice, modeling is one of the eight important and inter-related descriptions of what students might say or do as they learn math.

Is modeling just a “new and pretentious name for ‘word problems’ in the traditional sense” (Pollak 2011, p. vii)? Will it help us flip a familiar

paradigm (Lesh and Yoon 2007) away from telling students how important mathematics is in the world (“realizing mathematics”) toward students’ investigating a puzzling context for themselves (“mathematizing a reality”)? Could rich modeling tasks intrigue and involve students who had previously thought that mathematics was not meant for them? In this article, we describe the efforts of a classroom teacher and a teacher educator who worked to promote a better understanding of mathematical modeling for a group of urban middle school students.



SOLVING PROBLEMS AND MODELING

In typical problem-solving sequences, teachers tend to prepare students to solve problems in context, such as with word problems, by first teaching them skills, then facts, then big ideas (see **table 1**). Only after some degree of mastery of these isolated elements do students encounter problems in context, usually for the purposes of practice or application. Modeling differs noticeably in a number of

ways. First, modeling begins with the context and asks students to bring the mathematics they know to the problem. “Preteaching,” or guiding students toward specific mathematical content (e.g., skills, strategies, or models), runs counter to one purpose of modeling: allowing students to construct new mathematical understanding.

Lesh and Doerr (2003) suggest that the mathematics of modeling, unlike typical problem solving, moves “beyond short answers to narrowly

specified questions” (p. 3). The end result, a model, is not simply a single solution but a sharable, flexible, and reusable tool. It can be used to describe, explain, and/or predict a mathematical system or situation. It is common for more than one valid model to emerge from the same task.

Pollak (2011) describes modeling as the act of creating an “idealized version of a real-life situation” that has been “translated into mathematical terms” (p. vi). For example, in the Footprint problem (Koellner-Clark and Lesh 2003), students were not asked to find the height of a particular person whose footprints were left behind, but to develop a “how to” tool kit that could be generalized for any footprint (see **fig. 1**). In this way, modeling is an iterative activity that begins with a situation (sometimes not particularly mathematical at all) and leads to the formulation of a mathematical question or problem. The “solution” to this question or problem enables us to better understand not only the original situation but also other situations related to it (see **fig. 2**).

BEGINNING OUR JOURNEY

I met my coauthor, Meredith D. Lorber, several years ago when she was a graduate student (and first-year teacher) in a course I taught at Pace University. We have collaborated on several projects centered on our shared love of middle school students and the math they study. Lorber teaches at a diverse urban middle school where 65 percent of the students are African American, 11 percent Hispanic, 17 percent white, 4 percent Asian, and 3 percent other. Twenty percent of the students receive special education services, and 64 percent of students are eligible for a free or reduced-price lunch.

In this most recent stage of our partnership, our goal was twofold:

Table 1 Traditional problem solving differs from modeling in several discernible ways.

| Traditional Problem Solving | Modeling |
|--|--|
| Master prerequisite skills and ideas in a decontextualized setting. | Make mathematical sense of a problem while developing a sensible solution. |
| Practice skills on word problems. | Skills, ideas, and ability to solve problems are developed simultaneously during the process. |
| Learn problem-solving processes and heuristics (but not in reference to a particular problem). | What you need to solve the task is assumed to be in development (not mastered) prior to engaging with the problem. |
| House all these skills, ideas, and heuristics in a messy real-life situation (an applied problem). | Provide a messy, but accessible, context. |
| The end product is an answer or solution. | The end product is a generalizable model. |

Source: Adapted from Lesh and Doerr (2003)

Fig. 1 The Footprint problem is not superficial and relates to the mathematics, which made it a good choice for the investigation.

Early this morning, Shirley Jones, the local police detective, discovered that sometime late last night some nice people rebuilt the old brick drinking fountain in the park. The mayor, Maria Lopez, would like to thank the people who did it, but nobody saw who it was. All the police could find were lots of footprints. One of the footprints is shown here. But to find this person and his or her friends, it would help if we could figure out how tall he or she really is. Your job is to make a “how to” tool kit that the police can use to figure out how tall people are just by looking at their footprints. Your tool kit should work for the footprint shown here, but it should also work for other footprints.

Source: Koellner-Clark and Lesh (2003)

1. To give Lorber's 110 sixth-grade students the chance to model
2. To solidify our ideas about what makes mathematical modeling distinct and valuable

We started by looking for a good modeling task and modified the Footprint problem with Lorber's students in mind. With students gathered together at the front of the room, I shared the story of a recent school visit (see **fig. 3**).

Unlike typical problem solving, the context of modeling tasks should not be superficial or unrelated to the mathematics. Therefore, the task must be chosen or crafted carefully. Relocating the problem in an urban high school was our attempt to ensure that the context was accessible to students. A photograph of a large Converse All Stars shoe was revealed for a slightly different reason. Because the image was immediately familiar to almost all the students, the importance of this cultural artifact heightened students' intrigue.

Math educators are fond of debating the nuances of *real* and *real-world* contexts for students. We stay clear of this debate and side with Pimm (1997), who suggested that it is far more productive to find tasks that "offer the possibility of productive intellectual and emotional engagement" (p. xii). Our goal here was not to present a perfectly literal, real-world context, but rather one that felt imaginable, was believable, and was intriguing to Lorber's students. It is hoped that this task, with our guidance as teachers, would support them to do mathematics beyond what they already knew how to do.

MODELING: WHOSE QUESTIONS? WHOSE CHOICES?

Once the story had been presented and copies of the footprint were passed around, we paused to invite

students into the context. "I don't know about you," we noted, "but seeing this footprint raised a lot of questions for us. We are going to stop and ask you all to write down any questions you have about the person

wearing this shoe or perhaps about the situation itself. What do you want to know? What does this make you wonder about?"

Asking students to generate and share questions was a deliberate and

Fig. 2 This cycle describes the various steps in developing a model.

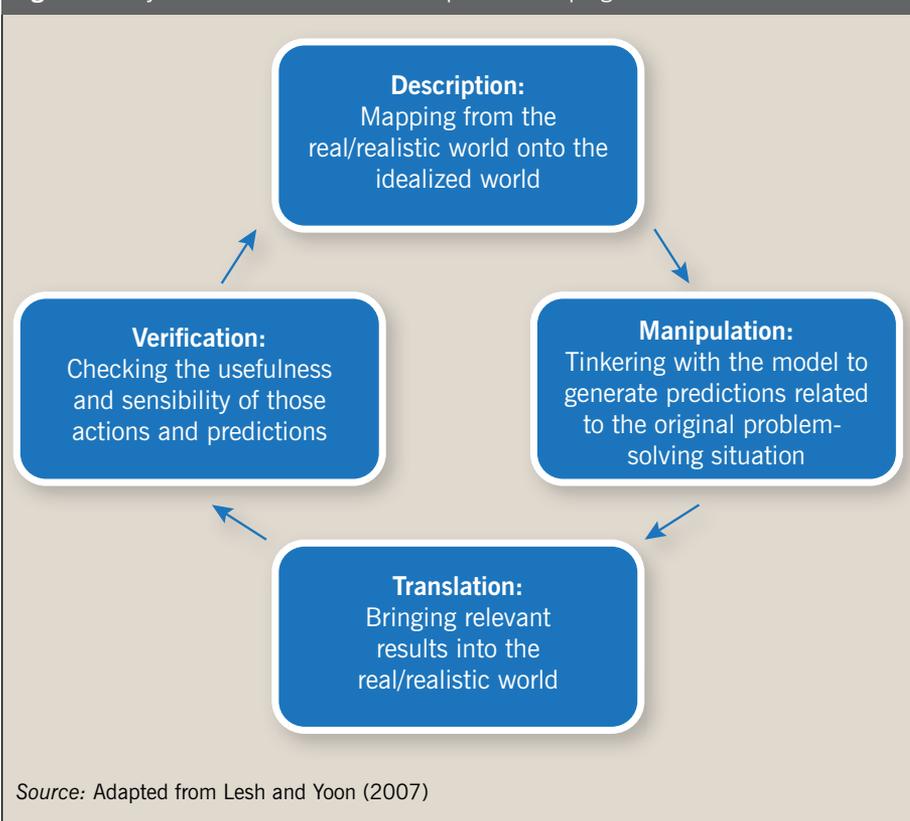


Fig. 3 Lorber's students heard information about a real-life experience.

I've been traveling to a bunch of schools, and I want to share with you something that I witnessed recently. Some of you know that we have some very large high schools, with two, three, even four thousand students in the same building. So, when I arrived at one of these schools the last few times, several adults and kids were talking about an alleged vandal who had been sneaking in at night after the security guards had gone home. Can someone remind us what a vandal is? And what does it mean to vandalize something?

So, we don't know much about this suspected vandal except what he or she left behind (messed up lockers, a broken-into chemistry lab, a torn up bulletin board). However, recently the vandal must have stepped in something on the way to or from the scene because this person left a footprint—a shoe print, more precisely. Now, with digital re-creation technology from the Department of Education, school security obtained an image of the shoe they believe this person was wearing. Because I happened to be visiting a teacher that day, I actually have a copy. Would you like to see it?

Table 2 Students looked at the footprint and formulated questions.

| What questions do you have about the alleged vandal? What does this situation make you wonder more about? | |
|---|--|
| Type of Question | Example |
| Motive | <i>Marcus</i> : "Why did they do it?" <i>Damali</i> : "I wonder why is the person vandalizing the school? What happened to the person that made him or her so mad?" <i>Henry</i> : "Were there multiple prints? Maybe that's a different shoe to throw off the police." |
| Physical aspects of the vandal | <i>Simone</i> : "I wonder how tall the person is." <i>Gym</i> : "How much do they weigh? How hard do they step?" <i>Tamaris</i> : "What does the culprit look like?" |
| Social and cultural aspects of the vandal (e.g., age, status, race, gender, and so on) | <i>Demarco</i> : "From that high school or just a citizen?" <i>Kayla</i> : "Was the person a girl or boy?" <i>Triniti</i> : "How many crimes did this person commit?" <i>Milo</i> : "Is he in a gang? Does he have a hard life? Is he in school?" |
| Details of the shoe | <i>Charity</i> : "I wonder if we can find that same design on people who have Converse." <i>Nzinga</i> : "What size is the shoe?" <i>Afia</i> : "What evidence can we find on the shoe?" <i>Judah</i> : "I wonder if the shoe print is slanted because they were running." <i>Kya</i> : "I think that the person had the shoes for a while because they are so dirty." |
| Details of the investigation | <i>Darius</i> : "How did they get the footprint?" <i>Eva</i> : "I wonder if you can look at all of the suspects' footprints." <i>Elijah</i> : "Are the police involved?" <i>Brandon</i> : "Did you ask the students who did it?" <i>Kelly</i> : "Is it possible to get a DNA sample from the shoe?" |
| The crime site | <i>Cinaya</i> : "Where did the security guards find the footprint?" |
| The acts of vandalism | <i>Elijah</i> : "Are they writing offensive things?" <i>Keith</i> : "What did the vandalism say?" <i>Marcus</i> : "Will they strike again?" <i>Bianca</i> : "Are there more than one perp?" <i>Caire</i> : "Is that even the shoe the suspect was wearing?" |
| Believability of the context | <i>Aaron</i> : "Don't they have cameras?" <i>Wittika</i> : "How can they have such a clear photo?" <i>YueYang</i> : "Why didn't the security catch them?" <i>Jade</i> : "Is this a real problem?" |

important move because the role of students' questions is critical in modeling contexts for several reasons. First, we wanted to encourage students to be curious and wonder aloud about this situation. Dedicating five full minutes to writing questions was an invitation to think about this context. Second, their questions served as a formative assessment of how they understood the situation. Some were troubled by the motives of the alleged vandal, others ques-

tioned the high school's own investigation surrounding the vandalism, and several others were intrigued by the shoe print itself (see **table 2**). As Nicol and Crespo (2005) have noted, when students are encouraged to visualize an imaginable situation and to become intrigued by it, a wider variety of genuine questions tend to emerge. Students' questions about context tend to differ markedly from the kind of contrived, predictable questions often heard in math classes

in which a single, correct answer is ultimately revealed by the teacher.

Finally, there remained an even more compelling reason for us to solicit students' questions. The heart of mathematical modeling, according to Pollak, is "problem finding before problem solving" (2011, p. vii). Identifying what is interesting, intriguing, and worth exploring about a situation is, in fact, a critical aspect of doing mathematics and one all too often not left up to students. When we tell

students what (and how) to solve or to prove, we miss entirely that identifying the problem or theorem was in fact the essence of the mathematical activity. We publicly listed many of the questions that students wanted to pursue (each interesting, valuable, and worthwhile), but then asked the class to consider which ones could be explored mathematically.

As a class, we decided to investigate the question “How tall is the vandal?” The question, as is, was not noticeably different from other school math problems. So we extended the question to ask, “Is it possible to determine the height of *any* person, based only on their shoe print? If so, how would you do it?” This subtle extension of the mathematics moved the task from mere problem solving (“How tall was the person wearing this shoe?”) toward modeling (“Does a model exist for predicting height from any shoe?”).

It was vital that students ask and answer their own questions throughout the process, not simply about the initial context. Next, we asked students to generate a list of items that they would need to investigate this situation. We compiled a class list and then reminded students of the tool table that was permanently located in one corner of Lorber’s room. Mindful that “even simple models involve making choices” (CCSSI 2010, p. 72), we wanted the choice of tool to be part of students’ decision-making process. Because tool choice and strategies are often linked, we witnessed several examples of students having to question, resolve, and negotiate these choices in their groups. Would they measure in inches or in centimeters? Could strips of large graph paper help them average the data? Would measuring the width of the shoe or maybe the perimeter be helpful? Could self-reported heights be trusted, or should students verify these claims?

Allowing students to make so



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many choices can lead to unnecessary confusion for them and a difficult-to-manage classroom for the teacher. To avoid these possibilities, teachers, textbooks, and even standardized tests often specify not only the tool to use but also the precise way to use a certain tool. We see this quite differently: When we keep students from making their own decisions, we deny them the chance to wrestle with important aspects of the mathematics, *and* we constrain the range of strategies that might emerge. It is not “unnecessary confusion” to ask students to choose

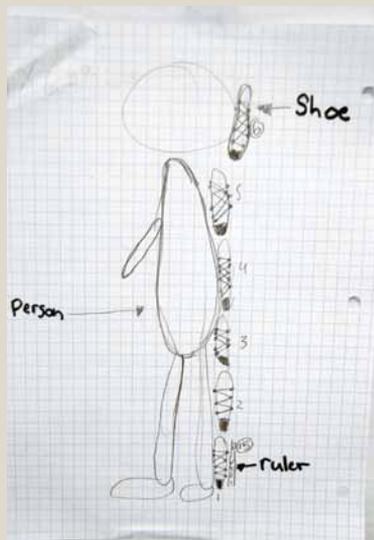
tools and strategies on their own. In fact, it is a critical aspect of constructing mathematics.

Our creation of a tool table was a direct result of our consideration of another Standard for Mathematical Practice: Use appropriate tools strategically (CCSSI 2010, p. 7). Although many students chose and used the tools that made sense to them, we were sometimes surprised by their choices. For example, we supplied graph paper, imagining a scatter plot of the data, not a double-bar graph with height-shoe length bars. We envisioned that string might be used to measure a person’s height, not the perimeter of the footprint. It was important to encourage students to take risks and sometimes struggle with using tools sensibly (e.g., taping a meterstick to the end of a yardstick) or to abandon tools for ones they found more useful.

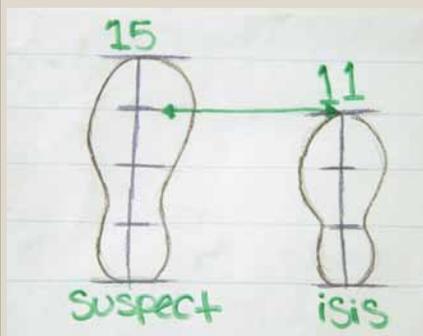
LEARNING MATHEMATICS THROUGH MODELING

In a traditional problem-solving sequence, problems in context rarely launch a unit because they are typically used to assess what students have learned. However, this is not the case with modeling. Because this activity was situated near the beginning of a unit on ratio and proportion, we

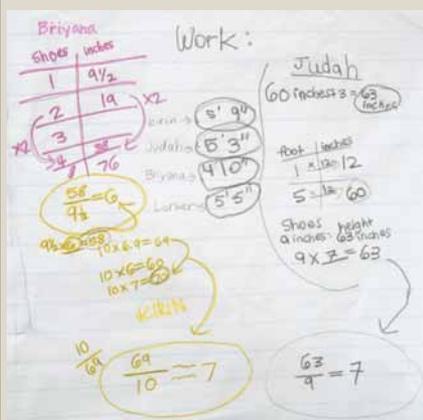
Fig. 4 Students applied different approaches to determine the suspect's height using a shoe as a unit of measure.



(a) A person's height, measured in shoes



(b) A comparison



(c) The suspect's shoe length and height in a ratio table



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Students were easily engaged in this activity because of the use of a familiar shoe.

agreed not to preteach skills or content, which might have ensured that all students emerged with the same understanding of the mathematics at hand. But that was not our purpose. We trusted that students would *learn mathematics through modeling* (Lesh and Caylor 2007), a perspective quite different than when contexts serve as a place to apply, practice, or assess the mathematics that students already know.

Students revealed that they were in different places, developmentally, with respect to their understanding of ratio and proportion. We knew that some students would struggle and that not all students would be thinking multiplicatively. It was clear to us, however, that every student understood the context and the mathematical question we decided to pursue together. Students wrestled with the idea about what it means to form a relationship between two quantities and how to represent and express that relationship. One group imagined a person's height as measured in shoes (see **fig. 4a**), another student argued that the ratio between her shoe and the suspect's shoe gave insight into the ratio between their heights (see **fig. 4b**), and several tried to distill the mathematical relationship between shoe length and height using a ratio table (see **fig. 4c**).

It is not “unnecessary confusion” to ask students to choose tools and strategies on their own. In fact, it is a critical aspect of constructing mathematics.

Meanwhile, others thought of the problem additively, noting the difference between the vandal's shoe length and their own, and did not see the problem multiplicatively at all (see **fig. 5**). We did not correct this misconception because Lorber's yearlong plan included returning to proportional reasoning a little later. Additionally, Lorber's classroom and school culture ensured that students would receive peer feedback (see **fig. 6**); whole-group feedback using a protocol; and, for those who continued working on the task, feedback from community members during a schoolwide event. Only when additive reasoning continued

to fail as an approach would students come to consider a different strategy and revise their thinking.

Students neither followed the same paths nor arrived at the same models. However, we witnessed the ways in which each student developed some new mathematics. Some saw a ratio table for the first time and began to try it out in other contexts. Others wrestled with ideas related to measurement and data collection. We even witnessed a contentious debate around precision: whether rounding data before or after they are averaged will make a significant difference. Experiences like these, we believe, are likely to build on the mathematics that students bring to a modeling task. Each student is learning something unique to their experiences, and every student is moving toward Lorber's yearlong goals for the class.

“THIS TIME, IT FELT DIFFERENT”

Like many students, Lorber's sixth graders have been solving “problems arising in everyday life, society, and the workplace” (CCSSI 2010, p. 7) for quite some time. But they told us that this time it felt different to them. Most students experience mathematics as being led down a fairly narrow path, toward some slice of mathematics that has been predetermined by a textbook, teacher, or test. Lorber's students got to theorize, express, and revise their own ways of thinking mathematically. A class conversation about whether or not a 7 foot 9 inch woman really existed, for example, allowed students to “interpret their mathematical results in the context of the situation” (CCSSI 2010 p. 7), not depend on their teacher to verify their results.

A different kind of thinking was required because “real-world situations are not organized and labeled for analysis” (CCSSI 2010, p. 72). Therefore, students were required to

Fig. 5 When students used additive thinking about known and unknown feet lengths, they were not corrected.

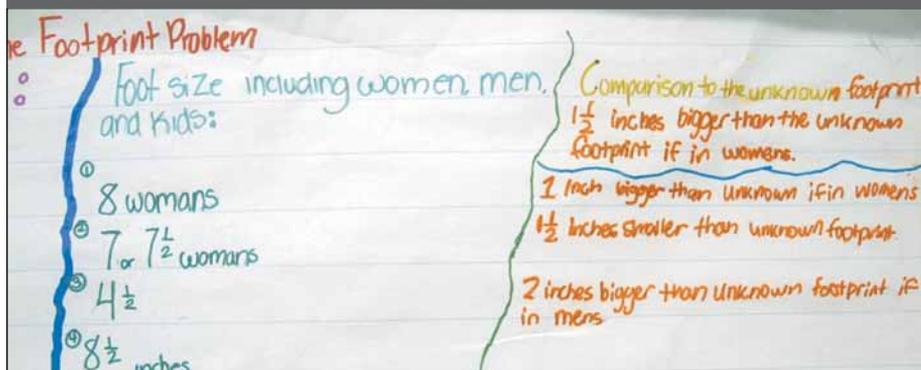


Fig. 6 This feedback format showed students how they would be evaluated.

PRESENTER FEEDBACK

Your name: _____ Presenting mathematician: _____

In line _____, where it says _____,

- I notice. . . .
- I would suggest. . . .
- I am confused because. . . .
- I think you did a good job of. . . .

My comment is most closely related to: (circle one)

| | | | |
|------------|-----------------------|----------|---------|
| Vocabulary | Mathematical examples | Accuracy | Clarity |
|------------|-----------------------|----------|---------|

formulate, sort, prioritize, and make sense of the problem at hand. When we think of modeling as a way to create mathematics, not simply as a place to apply the mathematics we currently possess, we shift students' notions from mathematics-as-procedure toward mathematics-as-interpretation (Lesh and Caylor 2007).

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I ♥ spherical analogs of truncated icosahedrons.

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